3. (b) Mr. Hazard demonstrated a rotating model for use in class instruction, in which the signs and values of sines and cosines are shown with their continuous variations throughout the entire circle.

5. Mr. Hutchinson's note dealt with real infinite series in a variable, x, which may be converted into power series in y by a substitution, $y=f(x)$. The interval of convergence of the power series is projected orthogonally onto the curve $y=f(x)$, thence orthogonally onto the x-axis, giving the regions of convergence, and divergence, of the original series.

8. There is always an extraneous factor when a differential equation of the type $f(x, y, p)=0$ has a singular solution. This paper by Professor Light shows how that factor can be found before formally solving the differential equation.

10. In a long article, Grundbegriffe der Mengenlehre, Neue Abhandlungen der Fries' schen Schule, 1906, G. Hessenberg raised the question as to what would happen if it could be proved, for example, that it is impossible to decide whether the number 2^{μ}, where $\mu=\sqrt{2}$ is an algebraic or a transcendental number. In his opinion mathematics would in this case be confronted with a difficulty more serious than any which has so far been encountered. The writer has not had access to the paper for many years and does not recall that any qualifying statements were made concerning the nature of the system of axioms used or concerning the laws of formal logic. In the absence of any such restrictions, it is easy to show that the difficulty would be solved by the introduction of two algebras, in one of which it would be an axiom that 2^{μ} is algebraic, in the other of which 2^{μ} would be transcendental.

11. In this paper, the author discusses some properties of those quartic surfaces with a double line which have the property that the conics on the surface, in pairs, constitute degenerate quartic curves of the first kind.

13. The projective differential properties of many figures may be studied by means of canonical expansions. Professor Stouffer derived such a canonical expansion for the equation of the plane curve, starting from a general expansion for the equation. The method required the proper choice of the triangle of reference. In a similar manner a canonical expansion suitable for the study of curved surfaces was also derived.

14. An answer to the question "What can the average student get out of the study of Mathematics" was given by Professor Adams. The values to be derived from the study were considered from the point of view of the student's social, economic, and individual life. His social life was shown to have been helped in giving him the valuable mental habits of neatness, orderliness, accuracy, persistence, intellectual honesty, and attention. His economic life has been benefited by direct application of mathematical principles to his business or profession, by fundamental training for further study in the sciences, and by the development of his ability to think clearly, rapidly, and accurately. His indi-

vidual life has profited by basic training in appreciation of beauty and by a realization of the exactness and order of the Universe.

PHILIP FITCH, *Secretary*

THE FIFTH MEETING OF THE INDIANA SECTION

The fifth meeting of the Indiana Section of the Mathematical Association of America was held May 11, 12, 1928 at Butler University, Indianapolis, Indiana.

There were sixty-one present at the meeting including the following thirty-one members of the Association: R. J. Aley, W. C. Arnold, Gladys L. Banes, Stanley Bolks, G. E. Carscallen, P. T. Copp, H. T. Davis, S. C. Davisson, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, H. E. H. Greenleaf, Laurence Hadley, Cora B. Hennel, F. H. Hodge, E. N. Johnson, Kathryn M. Kennedy, Florence Long, Juna M. Lutz, T. E. Mason, H. R. Mathias, R. E. Peterson, C. K. Robbins, D. A. Rothrock, J. R. K. Stauffer, C. E. Stout, K. P. Williams, H. E. Wolfe, W. A. Zehring, and H. A. Zinszer.

On Friday evening at 6:30 a banquet was held at the Claypool Hotel which was attended by members of the Association and their guests. President Aley of Butler University presided.

At eight o'clock a public lecture under the auspices of Butler University was given by President D. W. MOOREHOUSE of Drake University, Des Moines, Iowa, on the subject: "The Milky Way." President Moorehouse, by means of lantern slides, traced the ever interesting history of man's expanding knowledge of the universe. The speaker called special attention to the perplexing problem presented by the presence of black patches in the midst of brilliant star clouds and showed how the evidence points to the existence of great dark nebulous masses in the milky way.

At the session on Saturday morning on the Butler campus, presided over by Professor J. E. Dotterer, Manchester college, chairman, the following officers were elected: Professor H. E. H. GREENLEAF, De Pauw University, chairman; Professor H. A. ZINSZER, Hanover College, vice-chairman; Professor H. T. DAVIS, Indiana University, secretary-treasurer.

A chairman's address was made by Professor J. E. DOTTERER on the subject: "The Mathematician as a Salesman." Professor Dotterer pointed out the duty incumbent upon the teacher, in addition to his actual instruction, of showing the fundamental connection between mathematics and actual living. He urged the need of exhibiting mathematics not only as a tool used in solving and explaining the universe in which we live, but also as a discipline, a cultural subject and an art.

Professor R. H. COON of the Latin Department of Indiana University fur-

nished a pleasing variation to the program with a translation of Tycho Brahe's: "The Science of Mathematics and Astronomy," a public lecture given in the year 1574. The translation was prefaced by a summary of the life and achievements of the famous astronomer.

The morning program concluded with a pageant: "Pictures from the Pages of Mathematical History," which was presented by the department of mathematics of Butler University under the direction of Miss Gladys Banes, Professor E. N. Johnson, and Miss Juna Lutz. The words were written by Miss Lutz. The pageant comprised a survey of mathematical history from the time of Fuh-hi, 2852 B.C. to Henri Poincaré, and was especially noteworthy in the accuracy with which the various mathematicians had been reconstructed from their pictures.

The afternoon program consisted of the following papers:

1. "A generalization of the construction of the ellipse," by Professor F. H. HODGE, Purdue University.

2. "Some remarks concerning the exponential function," by Professor K. P. WILLIAMS, Indiana University.

3. "The teaching of analytic geometry," by Professor T. E. MASON, Purdue University.

4. "Function theoretic determination of the law of sines and the law of cosines for any plane triangle," by Mr. L. R. KELLAM, Indiana University. (Introduced by Professor Williams.)

5. "Some new formulas in curve fitting by least squares," by Mr. V. V. LATSHAW, Indiana University. (Introduced by Professor S. C. Davisson.)

6. "The place of statistics in the mathematics curriculum," by Professor H. T. DAVIS, Indiana University.

7. "On Fresnel's theory of diffraction," by Professor H. A. ZINSZER, Hanover college.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper generalized the method of constructing an ellipse by means of two concentric circles and the eccentric angle. One circle is taken with center at the origin and the second with center at $(h, 0)$. The locus of the point whose x and y intercepts are determined by the respective intersections of a straight line drawn from the origin through the two circles is found to be a cubic, some of whose properties were studied.

2. It is customary in books on the calculus to give only an incomplete discussion of $\lim_{n=\infty}(1+zn^{-1})^n$, leaving the problem to the theory of functions. On the other hand, books on function theory are likely to regard the problem as disposed of in the calculus. Professor Williams spoke on some of the details

necessary for a systematic treatment. Attention was also directed to certain generalizations that throw some light upon the exponential function.

3. Analytic geometry, according to Professor Mason, should be taught in such a manner as to give three things to the student: (1) A certain facility in drawing approximate graphs of functions; (2) ability to put into the form of an equation a description of a locus; (3) ability to interpret results. The learning of specific facts about particular curves is only incidental.

4. This paper showed how the law of sines and the law of cosines for any plane triangle may be determined from function theoretic properties. A simple list of these fundamental properties with specific application to one or two special cases for the determination of constants was used in obtaining the desired expressions.

5. Karl Pearson has indicated the desirability of computing the coefficients of polynomials fitted to data by the method of least squares where the values of the abscissa are assumed to be the integers from 1 to p. This paper gave these coefficients for the straight line, the parabola and the cubic. Coefficients were also computed for the quartic in the case where the number of points is odd.

6. It was pointed out in this paper that there seems to be a new movement toward the emphasis of statistics as a mathematical course. The theory of probability, which deals primarily with *a priori* probabilities, is admittedly a mathematical subject. Statistics which deals, on the other hand, with *a posteriori* probabilities obtained from empirical data, must not be approached with an aversion inherited from mathematical training in more subjective theories. The progress of this work in the mathematics department at Indiana University was explained.

7. The speaker gave a resumé of the Fresnel theory of diffraction with special reference to diffraction through a narrow slit bounded by parallel edges. The application of Cornu's spiral was explained.

At the conclusion of the meeting a vote of appreciation was extended to the department of mathematics of Butler University. The time and place of the next meeting were left in the hands of the executive committee.

H. T. DAVIS, *Secretary*

OBSERVATIONS ON SIMULTANEOUS QUADRATIC EQUATIONS[1]

By ARCHIBALD HENDERSON, University of North Carolina

1. Observations upon esthetic technic. It is the common practice of teachers of mathematics to consider their subject from a purely scientific, and often, alas! from an exclusively utilitarian point of view. In this paper the attempt is made to approach the problems treated from the esthetic standpoint. The mathematician, one finds, tends to make fetiches of brevity and uniformity, as other names for efficiency. Yet it is often the case that the sterotyped method, while reaching the sought for result in the briefest space of time, veils or ignores deeper, subtler, and more elegant properties associated with the problem treated. A straight line may be what is (inaccurately) designated the "shortest distance" between two points. It is, however, not necessarily the path which affords the widest horizon or the greatest beauties of the landscape.

It is not uncommon for mathematicians to comment admiringly upon the beauty or the elegance of a certain demonstration. How many of them would be able to describe lucidly the qualities which constitute the meaning of words of such rich content as beauty and elegance? The literature of mathematics is filled with examples of beauty and elegance in demonstration. The works of the classic French mathematicians in especial are characterized by these qualities. It is almost always the case that the supreme works of the greatest mathematicians are conspicuous for beauty and elegance, for an exceptional sense of balance and proportion, for symmetry and richness of content. Esthetic qualities tend to emerge most prominently in the greatest scientists. Nietzche goes so far as to assert that "the scientific man is the finest development of the artistic man." The classic exemplar is Leonardo da Vinci, in whose case no line of demarkation is to be observed between the artist and the scientist. An arresting contemporary illustration is the Theory of Relativity which, for all its scientific complexities, is essentially an artistic construct, marked by rare beauty, elegance, richness and variety. Professor Einstein once remarked to me that, after a certain high level of scientific technic is attained, science and art tend to coalesce in esthetic, plasticity and form. A vast field lies open to the investigator who would make a philosophical study of the esthetic qualities and characteristics of scientific technic, especially in the field of mathematics.

2. A particular case. The attempt is made here to treat, by known methods but with the esthetic aspect in mind, a familiar problem in simultaneous quadratic equations. I do not find this attractive treatment in the text

[1] Read at the twelfth annual meeting of the Mathematical Association of America, Nashville, December 30, 1927. I am indebted to my assistant, Mr. L. E. Bush, for aid in detail work and numerical computation, and for executing the figures.

books. The mathematician need not always go in sombre garb; he may sport a malacca cane and wear a gardenia in his buttonhole.

Consider the solution of the simultaneous system;

$$A:\begin{cases} x^2 + y = a^2 + b, & (1) \\ x + y^2 = a + b^2. & (2) \end{cases} \qquad (a \text{ and } b \text{ real})$$

If we proceed according to the usual stereotyped methods, we shall do one or the other of two things; eliminate one, or the other, of the variables from the two equations, solve the resulting quartic, and from these four solutions find by substitution the four values of the other variable: or plot the two parabolas (1) and (2), the co-ordinates of the intersections furnishing the plotted values to scale of the solutions common to equations (1) and (2).

Such a solution is skeleton, failing to embrace alternative algebraic systems of richer content and auxiliary associated configurations. There is lost to view the fundamental fact that the solution of a quartic may be made to depend upon the solution of a cubic equation: and that curves other than parabolas lend themselves more readily to quick and accurate plotting.

Making use of the two equations, which are readily derived from equations (1) and (2), namely

$$(x^2 - a^2)(y^2 - b^2) - (x - a)(y - b) = 0, \tag{3}$$

$$x^2 + y^2 + x + y - a(a + 1) - b(b + 1) = 0, \tag{4}$$

it is clear that the system A may be replaced by the point

$$x = a,\ y = b, \tag{5}$$

and any one of the following systems:

$$B:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x^2 + y = a^2 + b, & (1) \end{cases}$$

$$C:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x + y^2 = a + b^2, & (2) \end{cases}$$

$$D:\begin{cases} (x + a)(y + b) - 1 = 0, & (6) \\ x^2 + y^2 + x + y - a(a + 1) - b(b + 1) = 0. & (4) \end{cases}$$

It is to be noted that if any extraneous solution be introduced in any of these systems, it is rejected by the test of not satisfying equations (1) and (2). Thus from system D enters the extraneous solution $(-a-1,\ -b-1)$ which satisfies neither (1) nor (2). It may be noted in passing that the circle represented by

equation (4) is a member of the one-parameter family of conics passing through the points of intersection of the parabolas given by equations (1) and (2).

System B shows on its face that the solution of the problem, through the happy isolation of the single point (a, b), is reduced to the solution of a cubic, to wit

$$x^3 + ax^2 - (a^2 + 2b)x - (a^3 + 2ab - 1) = 0. \tag{7}$$

Obviously the ordinates of the three points are from symmetry or from system C given by the cubic equation

$$y^3 + by^2 - (b^2 + 2a)x - (b^3 + 2ba - 1) = 0. \tag{8}$$

From the practical standpoint, the graphical solution will not here be made to consist in the laborious plotting, by points, of the two parabolas given by the equations (1) and (2). The system D is chosen as supplying the two conics easiest to construct; the circle and the rectangular hyperbola referred to the asymptotes. For the circle, only the centre and radius are required: and these are given in this case by $(-1/2, -1/2)$ and $\frac{1}{2}[4a(a+1)+4b(b+1)+2]^{1/2}$. For the rectangular hyperbola, freehand plotting is quick and surprisingly accurate when only three points are plotted and the asymptotes, given here by $x+a=0$, $y+b=0$ are known.

The two parabolas given by the equations (1) and (2) cut the axes with respect to which they are severally symmetrical at (a^3+b) and $(a+b^3)$ respectively. Since (a, b) is always a solution, there must always be at least two real intersections. The problem resolves itself into four cases according to the nature of the intersections;

1. Four real and distinct.
2. Two real and coincident and two real and distinct.
3. Two real and distinct and two complex.
4. Two real and coincident and two complex.

As a numerical illustration, consider the case when $a=2$, $b=3$ (Fig. 1). Solving (7) by the usual trigonometric method for the solution of a cubic, and using (1) we obtain

$$x = \quad 3.1310, \quad -3.2829, \quad -1.8482,$$
$$y = -2.8030, \quad -3.7765, \quad 3.5920.$$

Therefore the solutions of (1) and (2) are

$$A:(2,3)\ ;\ B:(-1.8482, 3.5920)\ ;\ C:(-3.2829, -3.7765)\ ;\ D:(3.1310, -2.8030).$$

The extraneous solution introduced by system D, namely the point $E:(-3, -4)$, satisfies neither of equations (1) and (2).

3. The general case. The particular case treated above is a special type

which lends itself to the elegant method of solution adopted. This, however, is not the method employed in the treatment of the general case, which exhibits even greater elegance. Needless to say, the particular case may equally be solved according to the method now to be given: and this is done further on in this paper, in order to afford a comparison of methods.

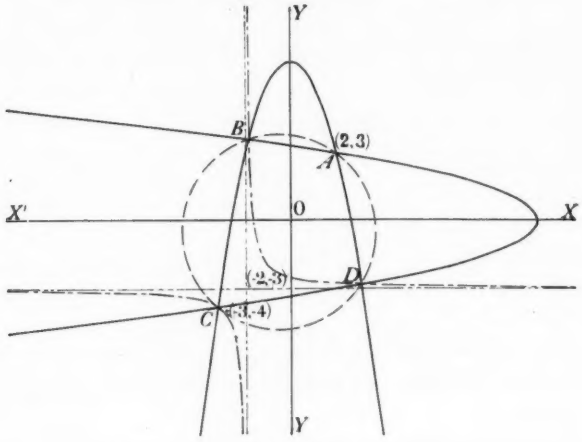


FIG. 1

Klein[1] has outlined a general program of dealing with the elementary mathematics from the higher standpoint. The solution here submitted for the solution of two simultaneous quadratic equations in two variables is, from the pedagogical point of view, a happy and forceful illustration of the Klein program.

Consider the equations

$$C_1: a_0x'^2 + a_1y'^2 + 2a_2x'y' + 2a_3x' + 2a_4y' + a_5 = 0, \tag{9}$$

$$C_2: b_0x'^2 + b_1y'^2 + 2b_2x'y' + 2b_3x' + 2b_4y' + b_5 = 0. \tag{10}$$

These equations represent in rectangular coordinates two conics which intersect in four points of some description, real or complex, rational or irrational, finite or ideal. Since some of the solutions may be infinite, we make use of homogeneous Cartesian coordinates, using the substitution

$$x' = x/z, \quad y' = y/z, \tag{11}$$

giving

[1] Felix Klein, *Elementarmathematik vom höheren Standpunkte aus*, 3te Auflage, Berlin, Springer. 1925.

$$C_1: a_0x^2 + a_1y^2 + 2a_2xy + 2a_3xz + 2a_4yz + a_5z^2 = 0, \tag{12}$$

$$C_2: b_0x^2 + b_1y^2 + 2b_2xy + 2b_3xz + 2b_4yz + b_5z^2 = 0. \tag{13}$$

On the four points of intersection of (12) and (13) there is a pencil (one-parameter family) of conics having the equation

$$C_1 + \lambda C_2 = 0 \tag{14}$$

or, written in full,

$$(a_0 + \lambda b_0)x^2 + (a_1 + \lambda b_1)y^2 + 2(a_2 + \lambda b_2)xy + 2(a_3 + \lambda b_3)xz + 2(a_4 + \lambda b_4)yz + (a_5 + \lambda b_5) = 0,$$

where λ is a parameter.

In this pencil of conics there will occur types of degenerate conics, namely certain pairs of straight lines. The condition that (14) be resolved into linear factors is

$$\begin{vmatrix} a_0 + \lambda b_0, & a_2 + \lambda b_2, & a_3 + \lambda b_3 \\ a_2 + \lambda b_2, & a_1 + \lambda b_1, & a_4 + \lambda b_4 \\ a_3 + \lambda b_3, & a_4 + \lambda b_4, & a_5 + \lambda b_5 \end{vmatrix} = 0. \tag{15}$$

This is a cubic equation in λ; and therefore in general we obtain three values of λ, i.e. $\lambda_1, \lambda_2, \lambda_3$. Each of these three values will cause equation (14) to break up into a pair of linear factors, representing geometrically a pair of straight lines. Denote by p_1 and p_2 the pair of lines arising from λ_1; by q_1 and q_2 the pair arising from λ_2; and by r_1 and r_2 the pair arising from λ_3.

Each of these pairs of lines lies on all four points of intersection of (12) and (13); and since it is impossible for a straight line to cut a conic in more than two points, each line separately passes through two of the points. Therefore on each point we have exactly three lines, one from the pair denoted by p, one from that denoted by q, and one from that denoted by r. Each of these three lines on any one point passes through one other point of the set. Therefore the lines constitute the pairs of opposite sides of a complete quadrangle on the four points, the p-lines constituting one pair of opposite sides, the q-lines another pair, and the r-lines the third (Fig. 2).

If the equation of any one of these lines be solved simultaneously with that of any other of the lines, with the single exception of that line with which it is paired, we obtain one of the required points of intersection of (12) and (13). The following table shows how all of the intersections are obtained, each in three different ways, from the equations of the six lines. Denote by $p_1 = 0$, the equation of p_1, etc. Then we have

$$A:\begin{cases} p_1 = 0 \\ q_1 = 0 \end{cases} \quad \begin{cases} p_1 = 0 \\ r_1 = 0 \end{cases} \quad \begin{cases} r_1 = 0 \\ q_1 = 0 \end{cases} \qquad p_1,\ q_1,\ r_1,$$

$$B:\begin{cases} p_1 = 0 \\ q_2 = 0 \end{cases} \quad \begin{cases} p_1 = 0 \\ r_2 = 0 \end{cases} \quad \begin{cases} q_2 = 0 \\ r_2 = 0 \end{cases} \qquad p_1,\ q_2,\ r_2,$$

$$C\begin{cases} p_2 = 0 \\ q_1 = 0 \end{cases} \quad \begin{cases} p_2 = 0 \\ r_2 = 0 \end{cases} \quad \begin{cases} r_2 = 0 \\ q_1 = 0 \end{cases} \qquad p_2,\ q_1,\ r_2,$$

$$D:\begin{cases} p_2 = 0 \\ q_2 = 0 \end{cases} \quad \begin{cases} p_2 = 0 \\ r_1 = 0 \end{cases} \quad \begin{cases} r_1 = 0 \\ q_2 = 0 \end{cases} \qquad p_2,\ q_2,\ r_1.$$

$A, B, C,$ and D above are the four points, and the last column shows the three lines on each point.

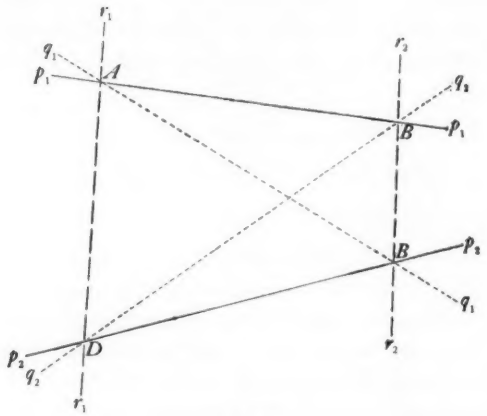


FIG. 2

If any solution gives the coordinate z the value zero, the point in question is infinite. Otherwise we may obtain the non-homogeneous coordinates by dividing x and y by the value of z, thus obtaining the solutions of (9) and (10).

A number of special cases arise according to the collocation of the four points of intersection. The case of three or four distinct points being collinear does not arise, since a straight line cannot cut a conic in more than two points.

CASE I. *Two intersections coincide.* Suppose, for example, that the intersections A and B coincide (Fig. 2). Then p_1 becomes tangent to the two given conics; and the two coincident degenerate conics, the pairs of straight lines $r_1r_2=0$ and $q_1q_2=0$ must eventuate from the same value of λ. Hence equation (15) must have a double root, which however does not make the left side of equation (14) a perfect square. Here the intersections are found by making $p_1=0$ and $p_2=0$ each simultaneous with either $q_1=0$ ($r_2=0$) or $q_2=0$ ($r_1=0$).

CASE II. *The intersections coincide in pairs.* Suppose, for example, that A and B coincide, and also that C and D coincide. Then p_1 and p_2 are both tangent to both conics; the former the common tangent at A, B: the latter the common tangent at C, D. Here the conics $r_1r_2=0$, $q_1q_2=0$ are coincident with each other, and in addition each is a pair of coincident lines. Hence equation (15) must have a double root which makes equation (14) a perfect square. In this case, the intersections may be found by making $q_1=0$ ($q_2=0$, $r_1=0$, $r_2=0$) simultaneous with $p_1=0$ and $p_2=0$.

CASE III. *Three intersections coincide.* Suppose, for example, that A, B, and C coincide. Then one line out of each pair becomes the common tangent to the two conics at A, B, C; while the other line in each pair coincides with the line joining A, B, C with D. Since the three degenerate conics are identical, equation (15) has a triple root which does not make equation (14) a perfect square. In this case, the intersections are found by making both $p_1=0$ ($q_1=0, r_2=0$) and $p_2=0$($q_2=0$, $r_1=0$) simultaneous with either one of the original conics given by the equations (12) and (13).

CASE IV. *Four intersections coincide.* In this case all six of the lines coincide with the common tangent to the two conics at the intersection. Hence equation (15) has a triple root, which makes equation (14) a perfect square. In this case, the single intersection (point of tangency) is found by making the equation $p_1=0$($p_2=0$, $q_1=0$, $q_2=0$, $r_1=0$, $r_2=0$) simultaneous with either one of the original conics given by the equations (12) and (13).

CASE V. *One of the original conics proper, the other degenerate.* If C_1 is proper and C_2 degenerate, then equation (15) reduces to a quadratic equation, since one value of λ must give rise to the equation $C_2=0$ when substituted in (14). If C_1 is degenerate and C_2 is proper, then equation (15) has one root zero, since one value of λ must give rise to the equation $C_1=0$ when substituted in equation (14). In this case we have only to make each of the lines of the degenerate conic simultaneous with the other conic, to find the intersections.

CASE VI. *Both original conics degenerate, with non-coincident meets.* Denote the two conics by $l_1l_2=0$, $m_1m_2=0$, and note that their meets (l_1, l_2), (m_1, m_2) are not coincident. Then since equation (15) must give rise to both C_1 and C_2, it must be a quadratic having one root zero. In this case the four intersections are found by making $l_1=0$ and $l_2=0$ simultaneous with $m_1=0$ and $m_2=0$.

CASE VII. *Both original conics degenerate, with coincident meets.* In this case equation (15) vanishes identically. To find the meet of the four lines make the equation of any of the four lines simultaneous with that of any of the other three.

Illustrations. To find the points of intersection of the two conics

$$C_1: 3x^2 + 5y^2 - 2xy + x - 5y - 10 = 0, \tag{16}$$

$$C_2: x^2 - y^2 - 6xy + 3x + y + 2 = 0. \tag{17}$$

The roots of (15) are $\lambda = 5, 1, -7/5$: and the corresponding line-pairs are

$$x(x - 4y + 2) = 0, \quad (x - y - 1)(x - y + 2) = 0,$$
$$(x + 2y - 4)(x + 2y + 2) = 0.$$

Solving these six linear equations in pairs simultaneously, never solving the members of any one pair with each other, we find the solutions of the original pair of equations to be $(0, -1), (0, 2), (2, 1), (-2, 0)$ (Fig. 3).

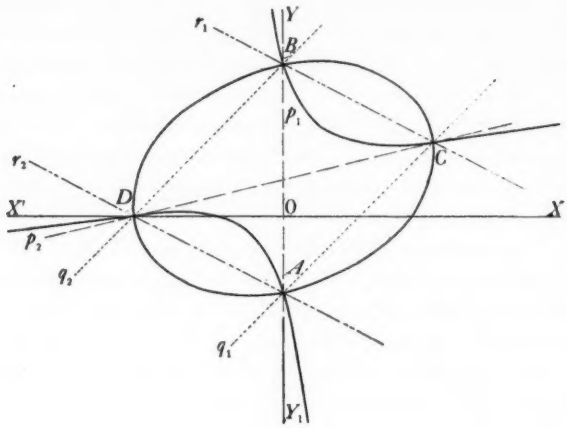


FIG. 3

As a second illustration, let us take the example of section 2:

$$C_1: x^2 + y = 7, \quad C_2: x + y^2 = 11.$$

Equation (15) is $\lambda^3 + 44\lambda^2 + 28\lambda + 1 = 0$, the roots of which, by either Newton's or Horner's method, after isolating the roots by the use of Sturm's functions, are found to be $-.038$, $-.608$, -43.355. Corresponding to these three values are the line-pairs

$$\left.\begin{aligned} p_1: x - .195y + 2.547 &= 0 \\ p_2: x + .195y - 2.585 &= 0 \end{aligned}\right\};$$

$$\left.\begin{aligned} q_1: x - .780y + .332 &= 0 \\ q_2: x + .780y - .940 &= 0 \end{aligned}\right\}; \qquad \begin{aligned} r_1: x - 6.584y - 21.583 &= 0 \\ r_2: x - 6.584y - 21.772 &= 0. \end{aligned}$$

Solving these lines in pairs, as indicated in the table below, we find the intersections of the two conics:

A	2	(p_2q_1)	(p_1r_2)	(q_1r_2)
		2.00	2.02	2.02
	3	2.99	3.00	3.00
		(p_1q_2)	(p_1r_2)	(q_2r_2)
B	−1.8482	−1.85	−1.82	−1.82
	3.5920	3.58	3.58	3.59
		(p_1q_1)	(p_1r_1)	(q_1r_1)
C	−3.2829	−3.28	−3.24	−3.24
	−3.7765	−3.79	−3.77	−3.77
		(p_2q_2)	(p_2r_1)	(q_2r_1)
D	3.1310	3.134	3.16	3.16
	−2.8030	−2.815	−2.80	−2.80

4. A method for the solution of the quartic equation, by adopting the device of causing a one-parameter family of conics to degenerate, has been given by Heilermann.[1] Consider the quartic equation

$$a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4 = 0. \tag{18}$$

Writing this in the form

$$a_0t^4 + 4a_1t^3 + 4a_2t^2 + 2a_2t^2 + 4a_3t + a_4 - 4a_0\theta t^2 + 4a_0\theta t^2 = 0, \tag{19}$$

and effecting the transformation

$$x = t^2, \quad y = 2t, \quad z = 1, \tag{20}$$

we have

$$(a_0x^2 + a_2y^2 + a_4z^2 + 2a_1xy + 2a_3yz + 2a_2zx) - a_0\theta(y^2 - 4zx) = 0. \tag{21}$$

Now the condition that the conic (21) shall degenerate into two straight lines is

$$\begin{vmatrix} a_0 & a_1 & a_2 + 2a_0\theta \\ a_1 & a_2 - a_0\theta & a_3 \\ a_2 + 2a_0\theta & a_3 & a_4 \end{vmatrix} = 0, \tag{22}$$

which is the familiar reducing cubic[2]

$$4a_0^3\theta^3 - a_0I\theta + J = 0, \tag{23}$$

where

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3.$$

[1] Zeitschrift für Mathematik und Physik, vol. 44 (1898).

[2] Burnside and Panton, *Theory of Equations*, p. 122.

By the use of this equation, it is not difficult to show that the equation (18), multiplied by a_0, breaks up into two factors;

$$[a_0t^2 + 2a_1t + a_2 + 2a_0\theta] \pm$$

$$(24) \qquad [2t\{a_1^2 - a_0a_2 + a_0^2\theta\}^{1/2} - \{(a_2 + 2a_0\theta)^2 - a_0a_4\}^{1/2}] = 0.$$

Solving (22) we obtain three values of θ, either of which leads to two quadratic equations (24), giving the roots paired in the three ways

$$\left.\begin{matrix}\alpha,\beta\\ \gamma,\delta\end{matrix}\right\}; \quad \left.\begin{matrix}\alpha,\gamma\\ \beta,\delta\end{matrix}\right\}; \quad \left.\begin{matrix}\alpha,\delta\\ \beta,\gamma\end{matrix}\right\}.$$

An application; Let us apply this method to the solution of the first illustration given in article 3, namely to find the coordinates of the points of intersection of the two conics

$$(16) \qquad 3x^2 + 5y^2 - 2xy + x - 5y - 10 = 0,$$

$$(17) \qquad x^2 - y^2 - 6xy + 3x + y + 2 = 0.$$

Using Sylvester's dialytic method of elimination, the ordinates of the four intersections of equation (16) and (17) are given by

$$(25) \qquad \begin{vmatrix} 3 & 1-2y & 5(y^2-y-2) & 0 \\ 0 & 3 & 1-2y & 5(y^2-y-2) \\ 1 & 3(1-2y) & -(y^2-y-2) & 0 \\ 0 & 1 & 3(1-2y) & -(y^2-y-2) \end{vmatrix} = 0,$$

which reduces to the equation

$$(26) \qquad y^4 - 2y^3 - y^2 + 2y = 0.$$

Employing the method of Heilermann outlined above, equation (22) furnishes three values of θ: $^7/_{12}$, $-^1/_6$, $-^5/_{12}$. The value $^7/_{12}$, for example, set in equation (24) gives the two equations (y for t):

$$y^2 + y = 0, \quad y^2 - 3y + 2 = 0,$$

furnishing the values $0, -1, 1, 2$. Solving for x from either (16) or (17) we obtain as the values of the coordinates, finally

$$A:(0,-1); \quad B:(0,2); \quad C:(2,1); \quad D:(-2,0)$$

This simple example is given to illustrate the method, although obviously equation (26) may be solved by inspection.

In conclusion, it is worthy of note that the present paper puts constantly in evidence the solution of the quartic equation as dependent upon the solution of a cubic equation.

ON THE EARLY HISTORY OF THE DECIMAL POINT

By JEKUTHIEL GINSBURG, Teachers College, Columbia University

The early history of the decimal fraction has been considered so extensively that it would seem that little could remain unsaid concerning its inventors. Even in the less important matter of symbolism, the ground would seem to have been covered, thanks to the investigations of men like Braunmühl, Cajori, Eneström, Glaisher, Karpinski, Smith, and Tropfke. As to the notion of the fraction itself, the facsimiles in Smith's *History of Mathematics* (Vol. 2 pp. 236–243, with the accompanying text) set forth the case with considerable completeness, showing that such fractions were used some time before the decimal point was suggested. The latter is merely one of several symbols used to separate the fractional and integral parts, and the question as to what device shall be selected is merely one relating to convenience.

The purpose of this article is to call attention to a fact which seems to have been overlooked by all historians, namely, that the credit of being the first to use a dot (point) as a separatrix, with a full knowledge of its significance in this connection, is due to the Jesuit father Christopher Clavius (1537–1612). In his work on the astrolabe, published in Rome in 1593 he gives (pp. 195–270) a "Tabula Sinuum," where the proportional parts are separated from the integers by periods (points, dots). A portion of the table is here shown. So far as now known, this is the first appearance of the decimal point in a work in which its full significance is given. The table antedates the Pitiscus edition of 1608, which has been so carefully studied by Professor Cajori.[1] It also antedates Napier's *Rhabdologia* (1617), Wright's translation of the *Mirifici logarithmorum canonis descriptio* (1616), and Kepler's *Ausszug auss der uralten Messe-Kunst Archimedis* (1616), in each of which the decimal point or comma is used. As to Bürgi's use of the symbol in a manuscript of 1592, the claim is no longer seriously considered.[2] The name of Stevin, who wrote the first book upon decimal fractions,[3] need not be considered in this connection since he made no use of the symbolism under discussion.

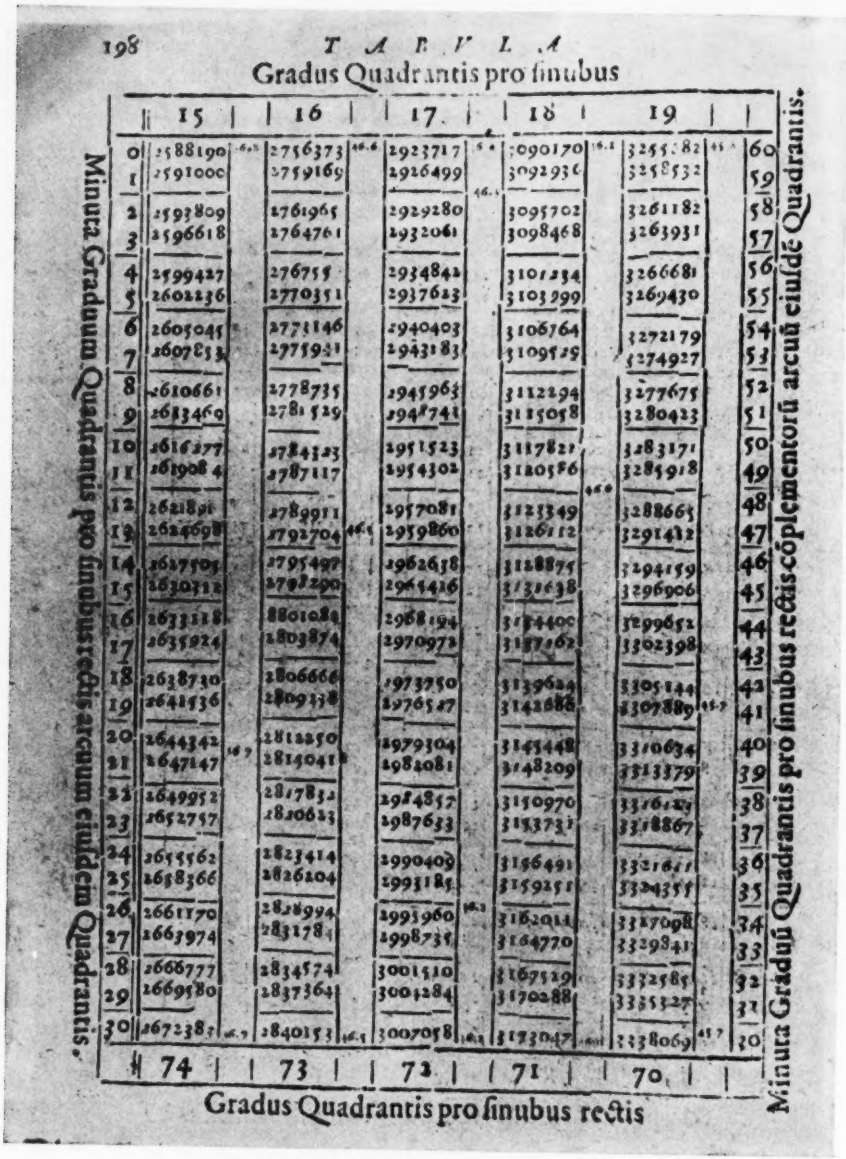
It remains to consider whether Clavius understood the significance of the symbol, for otherwise he would be entitled to but little more credit than Pellos who, in 1492, wrote 538694.3 as the quotient of 53836943 by 10, but showed no further appreciation of decimals. That Clavius actually did understand its significance appears by the following statement on page 228:

——quoniam inter duos sinus grad. 16 min. 12 grad. 16 min. 13 positi sunt duo binumeri 46.5 colligemus uni secundo inter minutum 12 13 gradus 16 congrue particulas $46\frac{5}{10}$, ex differentio 2793 inter duos sinus

[1] *Did Pitiscus use the decimal point?* In Archivio di storia della scienza, vol. 4 (1923), pp. 313–28.

[2] See L. C. Karpinski, *On the decimal point*, Science, (2), vol. 45 (1912), pp. 663–665.

[3] *Le Thiende*, Antwerp, 1535, with a French translation (*La disme*) in the same year.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TABVLA

Gradus Quadrantis pro ſinubus

	15		16		17		18		19		
0	2588190	46.3	2756373	46.6	2923717	46.9	3090170	46.1	3255782	45.4	60
1	2591000		2759169		2926499		3092936		3258532		59
2	2593809		2761965		2929280	46.1	3095702		3261282		58
3	2596618		2764761		2932061		3098468		3263931		57
4	2599427		276755		2934842		3101234		3266681		56
5	2602236		2770351		2937623		3103999		3269430		55
6	2605045		2773146		2940403		3106764		3272179		54
7	2607853		2775941		2943183		3109529		3274927		53
8	2610661		2778735		2945963		3112294		3277675		52
9	2613469		2781529		2948741		3115058		3280423		51
10	2616277		2784323		2951523		3117821		3283171		50
11	2619084		2787117		2954302		3120586		3285918		49
12	2621891		2789911		2957081		3123349	46.0	3288665		48
13	2624698		2792704	46.1	2959860		3126112		3291412		47
14	2627505		2795497		2962638		3128875		3294159		46
15	2630312		2798290		2965416		3131638		3296906		45
16	2633118		2801082		2968194		3134400		3299652		44
17	2635924		2803874		2970972		3137162		3302398		43
18	2638730		2806666		2973750		3139924		3305144		42
19	2641536		2809458		2976527		3142686		3307889	45.7	41
20	2644342		2812250		2979304		3145448		3310634		40
21	2647147	46.7	2815041		2982081		3148209		3313379		39
22	2649952		2817832		2984857		3150970		3316123		38
23	2652757		2820623		2987633		3153731		3318867		37
24	2655562		2823414		2990409		3156491		3321611		36
25	2658366		2826204		2993185		3159251		3324355		35
26	2661170		2828994		2995960	46.3	3162011		3327098		34
27	2663974		2831784		2998735		3164770		3329841		33
28	2666777		2834574		3001510		3167529		3332585		32
29	2669580		2837364		3004284		3170288		3335327		31
30	2672383	46.7	2840153	46.1	3007058	46.3	3173047	46.0	3338069	45.7	30
	74		73		72		71		70		

Gradus Quadrantis pro ſinubus rectis

Minuta Graduum Quadrantis pro ſinubus rectis arcuum eiuſdem Quadrantis.

Minuta Graduũ Quadrantis pro ſinubus rectis cõplementorũ arcuũ eiuſdẽ Quadrantis.

Facsimile of p. 198 of the "Astrolabe" of Clavius (1593)

2789911. 2792704praedictorum arcuum grad. 16 min. 12 et grad 16min. 13 qua to ta differentia secundia 60, Hoc Est.

It amounts to the following:

In the table on p. 198, the number 46.5 is put between sin 16°12′ and 16°13′. Hence to every second (of the difference) between 16°12′ and 16°13′ we will apportion $46\frac{5}{10}$ particles of the difference (2793) between 2789911 (sin 16°12′) and 2792704 (sin 16°13′) etc.

It would seem, therefore, that up to the present time the evidence is conclusive that Clavius was the first to use the decimal point with a clear understanding of its significance. This statement does not, of course, invalidate the positive evidence that the decimal fraction was used with other symbols long before his time.

PROPERTIES OF TWO POINTS ASSOCIATED WITH A TRIANGLE

By J. H. WEAVER, Ohio State University

The Brocard points of a triangle are defined as the intersections of two triads of circles described on BC, CA, AB as chords in such a way that the external segments of the circles contain the angles (1) C, A, B, and (2) B, C, A respectively.[1]

If the triads of circles are drawn in such a way that the internal segments contain the angles (1) C, A, B and (2) B, C, A, respectively, two other points P_2 and P_4 will be determined. It is the purpose of the present paper to discuss some of the properties of the points P_2 and P_4.

THEOREM I: *The pedal triangles of P_2 and P_4 are similar.*

PROOF: Since in the figure P_2 lies within the angle BCA and $D_2E_2F_2$ is the pedal triangle of P_2 we have $\angle BP_2C = \angle(A+D_2)$.[2] Therefore $\angle D_2 = \angle(C-A)$. Similarly $\angle E_2 = \angle(A-B)$; $\angle F_2 = \angle(\pi+B-C)$. Also if, $D_4E_4F_4$ is the pedal triangle of P_4 it may be shown in the same way that

$$\angle D_4 = \angle(A-B), \quad \angle E_4 = (\pi+B-C), \quad \angle F_4 = \angle(C-A).$$

Hence the pedal triangles of P_2 and P_4 are similar.

---

[1] For a discussion of the various definitions of Brocard points and their properties see McClelland, *Geometry of the Circle*, London, 1891, pp. 60–70; Court, *College Geometry*, Richmond, 1925, pp. 24–38; Gallatley, *Modern Geometry of The Triangle*, London, pp. 94–108; Coolidge, *A Treatise on The Circle and the Sphere*, Oxford, 1916, pp. 60–84; *Encyclopädie der Mathematischen Wissenschaften*, Band III_1, Heft 7.

[2] See Gallatley, *Modern Geometry of the Triangle*, p. 37.

THEOREM II: *The three pairs of lines BP_2 and CP_4, CP_2 and AP_4, AP_2 and BP_4 are parallel.*

PROOF: It is shown in Gallatley's, *Modern Geometry of the Triangle*, p. 42, that the trilinear coordinates of the points P_2 and P_4 may be written as follows: for P_2,

$$\alpha:\beta:\gamma = \frac{\sin C}{\sin (C-A)}:\frac{\sin A}{\sin (A-B)}:\frac{\sin B}{\sin (B-C)};$$

for P_4,

$$\alpha:\beta:\gamma = \frac{\sin B}{\sin (A-B)}:\frac{\sin C}{\sin (B-C)}:\frac{\sin A}{\sin (C-A)};$$

where AB is the line $\gamma=0$, BC is the line $\alpha=0$ and CA is the line $\beta=0$. The equation of AP_2 is

$$\frac{\beta \sin B}{\sin (B-C)} - \frac{\gamma \sin A}{\sin (A-B)} = 0;$$

and the equation of BP_4 is

$$\frac{\alpha \sin A}{\sin (C-A)} - \frac{\gamma \sin B}{\sin (A-B)} = 0.$$

Now the angle which any line $l\alpha+m\beta+n\gamma=0$ makes with the line $\gamma=0$ is given by the relation[1]

$$\tan \psi = \frac{l \sin B - m \sin A}{l \cos B + m \cos A - n}.$$

For the line AP_2 this relation becomes

$$\tan \psi_2 = \frac{\sin A \sin B \sin (A-B)}{\sin B \cos A \sin (A-B) + \sin A \sin (B-C)};$$

and for BP_4 it becomes

$$\tan \psi_4 = \frac{\sin A \sin B \sin (A-B)}{\sin A \cos B \sin (A-B) + \sin B \sin (C-A)}.$$

The values for $\tan \psi_2$ and $\tan \psi_4$ are equal; hence the lines AP_2 and BP_4 are parallel. Similarly it may be shown that CP_2 is parallel to AP_4 and that BP_2 is parallel to CP_4.

---

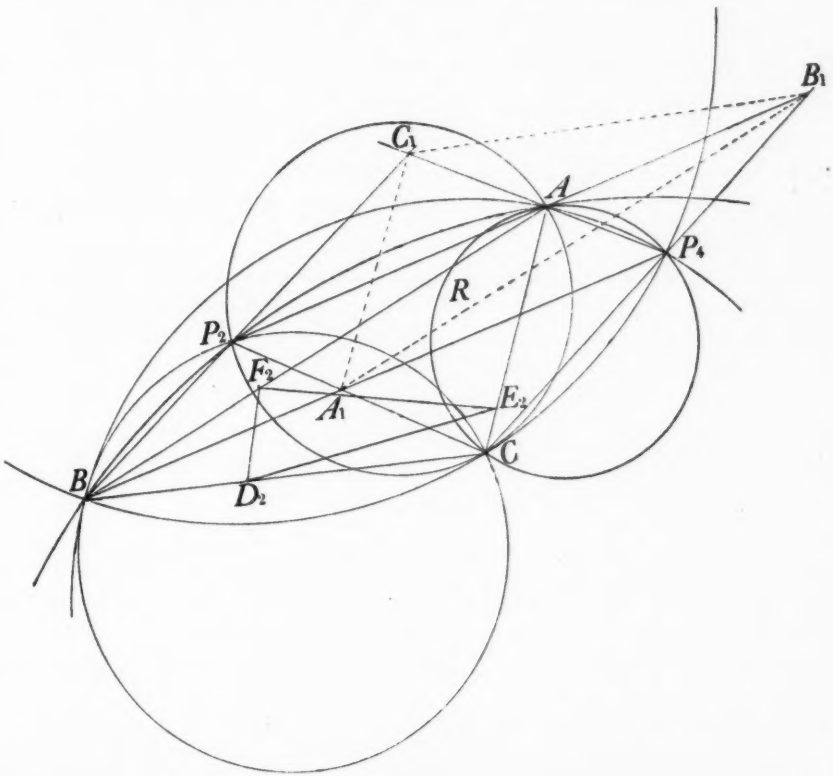
[1] See Whitworth, *Modern Analytic Geometry*, Cambridge, 1866, p. 49.

Let BP_4 and CP_2 intersect in A_1, AP_2 and CP_4 intersect in B_1, and BP_2 and AP_4 intersect in C_1. We may then prove

THEOREM III: *The triangles ABC and $A_1B_1C_1$ are congruent and have the sides AB, BC, CA respectively parallel to A_1B_1, B_1C_1, C_1A_1.*

PROOF: Without much difficulty it may be shown that the equation of A_1C_1 is

$$\begin{aligned}
&\alpha[\sin^2 A \sin^2 B \sin^3 (C - A) \sin (B - C) \\
&\quad - \sin A \sin B \sin^2 C \sin^2 (C - A) \sin^2 (A - B)] \\
&+ \beta[\sin^2 A \sin^2 C \sin^2 (A - B) \sin^2 (B - C) \\
&\quad - \sin A \sin^2 B \sin C \sin (A - B) \sin^2 (C - A) \sin (B - C)] \\
&+ \gamma[\sin^2 B \sin^2 C \sin (A - B) \sin^3 (C - A) \\
&\quad - \sin^2 A \sin B \sin C \sin^2 (B - C) \sin^2 (C - A)] = 0.
\end{aligned}$$



This line will be parallel to the line $\beta = 0$[1] if

$$\sin^2 A \sin^2 B \sin C \sin^2 (C - A) \sin (B - C)$$
$$- \sin A \sin B \sin^3 C \sin^2 (C - A) \sin^2 (A - B)$$
$$- \sin A \sin^2 B \sin^2 C \sin (A - B) \sin^3 (C - A)$$
$$- \sin^3 A \sin B \sin C \sin^2 (B - C) \sin^2 (C - A) = 0.$$

This is true. Therefore the line A_1C_1 is parallel to AC, and since A_1C_1 and AC are parallels between parallels, we have $A_1C_1 = AC$. A similar relationship holds for the pairs of lines B_1C_1 and BC and A_1B_1 and AB. Hence the theorem.

The following theorem will be evident from a glance at the figure.

THEOREM IV: *The triangles ABC and $A_1B_1C_1$ are homothetic, the homothetic center being R, the mid-point of P_2P_4.*

THEOREM V: *The two points P_2 and P_4 bear the same relationship to each of the triangles ABC and $A_1B_1C_1$.*

PROOF: This follows from the fact that since R is the homothetic center of the two triangles, P_2 referred to the triangle ABC corresponds to P_4 referred to the triangle $A_1B_1C_1$.

THEOREM VI: *The two triangles ABC and $A_1B_1C_1$ are triply perspective, the centers of perspective being the three collinear points P_2, R, P_4, and the three axes being the line at infinity and two parallel lines symmetrically situated with respect to R.*

PROOF: A glance at the figure will show the first part of the theorem; and since when two triangles are perspective from three collinear points they will be perspective from three concurrent lines, and since one of these lines is the line at infinity, the other two will be parallel.

A PROJECTIVE THEOREM ON THE PLANE PENTAGON

By EDWARD KASNER, Columbia University

This paper is concerned with a projective theorem relating to the plane pentagon. In order to be able to state it, we first define two projective constructions, the diagonal construction D and the inscribed construction I, by means of which two new pentagons are derived from the original one.

Let A, B, C, D, E be the corners of the given pentagon P. The construction D then consists in drawing the five diagonals BD, CE, DA, EB, AC of P, intersecting every pair of consecutive ones in the indicated order and taking

[1] See Whitworth, *Modern Analytic Geometry*, p. 49.

these points of intersection (in the same order) for the corners of a new pentagon.[1] The construction I consists in finding the points of contact of the conic inscribed to P with the sides CD, DE, EA, AB, BC of P and taking these points in the order of the sides of P which they lie on for the corners of a new pentagon.[2]

We term the two pentagons derived in this manner from the original one, P, the *diagonal* and the *inscribed pentagon* of P and abbreviate them by $\mathsf{D}P$ and $\mathsf{I}P$.

Our theorem now reads:

For any plane pentagon P the inscribed pentagon of its diagonal pentagon and the diagonal pentagon of its inscribed pentagon are identical. In formulas,

$$\mathsf{ID}P = \mathsf{DI}P.$$

Differently expressed: *the diagonal construction* D *and the inscribed construction* I *are commutative.*

Let the corners of $\mathsf{D}P$ be called $A', B', \cdots$ so that

$$A' = (BD)(CE)^{3}, \qquad B' = (CE)(DA), \cdots, \tag{1}$$

and the corners of $\mathsf{I}P$ be called $A_1, B_1, \cdots$ so that they lie on the sides $CD, DE, \cdots$. Then according to Brianchon's theorem

$$A_1 = (AA')(CD)^{4}, \qquad B_1 = (BB')(DE), \cdots. \tag{2}$$

With the same method of notation, superscripts and subscripts being used to denote the diagonal and the inscribed pentagon of a given pentagon, the corners $(A')_1, (B')_1, \cdots$ of $\mathsf{ID}P$ and the corners $(A_1)', (B_1)', \cdots$ of $\mathsf{DI}P$ are determined as follows:[5]

$$\begin{aligned} &\text{(a)} \quad (A')_1 = (A'A'')(C'D')^{6}, \quad (B')_1 = (B'B'')(D'E'), \cdots, \\ &\text{(b)} \quad (A_1)' = (B_1D_1)(C_1E_1), \quad (B_1)' = (C_1E_1)(D_1A_1), \cdots, \end{aligned} \tag{3}$$

where

$$A'' = (B'D')(C'E'), \quad B'' = (C'E')(D'A'), \cdots. \tag{3}'$$

To prove our theorem, we must show that

$$(A')_1 = (A_1)'\,; \qquad (B')_1 = (B_1)'\,; \cdots.$$

---

[1] A', B', C', D', E' in the accompanying drawing.

[2] A_1, B_1, C_1, D_1, E_1, in the accompanying drawing.

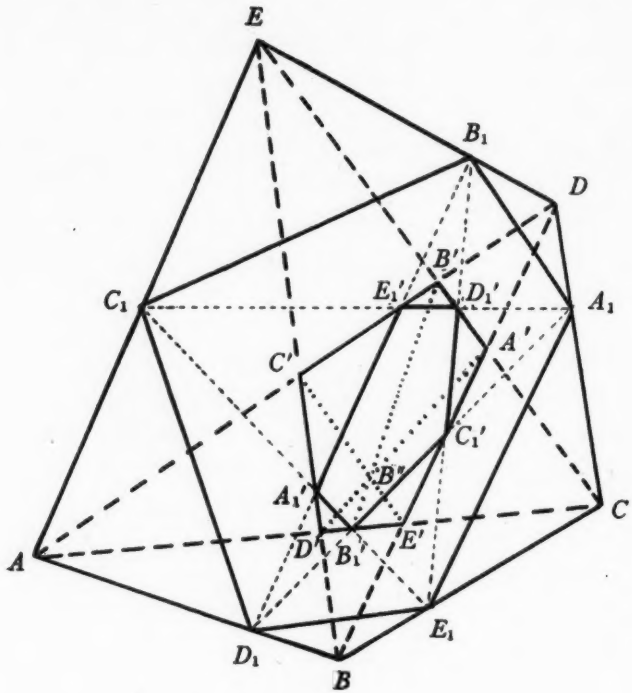
[3] We use this notation to indicate that the point on the left hand side is the point of intersection of the two lines in brackets on the right hand side.

[4] The lines $AA', BB', \cdots$ have not been put into the drawing, to avoid confusing the reader by too many lines.

[5] In the drawing we have indicated the coinciding corners of $\mathsf{ID}P$ and $\mathsf{DI}P$ by $A_1', B_1', \cdots$.

[6] In the drawing we have carried out the complete construction only for $(B')_1$.

On account of the absolute symmetry of construction for all five pairs of points, it will be sufficient to show the identity for any one pair, say $(B')_1$ and $(B_1)'$.



We give the proof that $(B')_1$ and $(B_1)'$ coincide analytically, and for this purpose introduce the special homogeneous coordinate system which has its fundamental points at the points, A, B and C and its point of unity at the point D. The coordinates of E in this system shall be denoted by a, b, 1.

In this system we find for the coordinates of the corners and sides of P,

$$
\begin{array}{llllll}
A\colon & 1 & 0 & 0 & CD\colon & 1 \quad -1 \quad 0 \\
B\colon & 0 & 1 & 0 & DE\colon & b-1 \quad 1-a \quad a-b \\
C\colon & 0 & 0 & 1 \ ; & EA\colon & 0 \quad -1 \quad b \ ; \\
D\colon & 1 & 1 & 1 & AB\colon & 0 \quad 0 \quad 1 \\
E\colon & a & b & 1 & BC\colon & 1 \quad 0 \quad 0
\end{array} \tag{4}
$$

the sides and corners of $\mathsf{D}P$, according to (1),

$$
\begin{array}{lllllllll}
A'D' = EB: & -1 & 0 & a & A': & a & b & a & \\
D'E' = AC: & 0 & 1 & 0 & B': & a & b & b & \\
E'A' = BD: & 1 & 0 & -1 & ; \quad C': & a & 1 & 1 & ; \\
A'B' = CE: & -b & a & 0 & D': & a & 0 & 1 & \\
B'C' = DA: & 0 & -1 & 1 & E': & 1 & 0 & 1 &
\end{array} \tag{5}
$$

the lines AA', BB', $\cdots$, according to (4) and (5),

$$
\begin{array}{llll}
AA': & 0 & -a & b \\
BB': & b & 0 & -a \\
CC': & -1 & a & 0 \quad ; \\
DD': & -1 & 1-a & a \\
EE': & b & 1-a & -b
\end{array} \tag{6}
$$

the corners of $\mathsf{I}P$, according to (2), (4) and (6),

$$
\begin{array}{lccc}
A_1: & b & b & a \\
B_1: & a(1-a) & b^2 - 2ab + a & b(1-a) \\
C_1: & ab & b & 1 \\
D_1: & 1-a & 1 & 0 \\
E_1: & 0 & b & 1-a
\end{array} \tag{7}
$$

To find $(B')_1$ we first calculate the coordinates of B'' according to (3)′ and (5).

$$B'': \quad a(b+1) \quad b \quad a+b.$$

Then, according to (3a) and (5), the coordinates of $(B')_1$ are

$$(B')_1: \quad b \quad 0 \quad 1.$$

The coordinates of $(B_1)'$ are immediately obtained, according to (3b), from (7) as

$$(B_1)' : b \quad 0 \quad 1$$

and are thus equal to those of $(B')_1$.

In this manner we have shown that the points $(B_1)'$ and $(B')_1$ coincide. As stated in the beginning of the proof, it follows by virtue of symmetry that also the remaining points $(A_1)'$, $(C_1)'$, $(D_1)'$, $(E_1)'$ coincide with the points $(A')_1$, $(B')_1$, $(D')_1$, $(E')_1$ respectively. Thus our theorem is proved:

The pentagon obtained by first applying to the given pentagon P the diagonal construction D *and then to the pentagon* $\mathsf{D}P$ *the inscribed construction* I, *is the*

same as the pentagon obtained by first applying to P the inscribed construction I *and then to* IP *the diagonal construction* D.

This theorem was first stated by the author in 1896 at a students' colloquium in Columbia University. It was published (among other results) in an abstract in the Bulletin of the American Mathematical Society, vol. 9 (1903), pp. 397, 398. Ten years later Professor Bateman, to whom I communicated the theorem, indicated a synthetic metrical proof. This year, three members of my seminar, Miss Hofmann, Mr. Donahue, and Mr. Feld, worked out synthetic projective proofs.

Among the corollaries stated in the Bulletin abstract, the most interesting is this (valid for at least convex pentagons):

The limit point of the sequence of successive inscribed pentagons coincides with the limit point of the sequence of successive diagonal pentagons.

SYNTHETIC PROOF OF PROFESSOR KASNER'S PENTAGON THEOREM

By LULU HOFMANN, Columbia University

This paper gives a synthetic proof of the pentagon theorem stated and proved analytically by Professor Kasner in the preceding article. We will not re-explain the notions of the diagonal and inscribed pentagon, DP and IP of a given pentagon P nor the notation used by Professor Kasner. We merely repeat the theorem:[1]

For any plane pentagon P the inscribed pentagon IDP *of its diagonal pentagon* DP *and the diagonal pentagon* DIP *of its inscribed pentagon* IP *are identical*:

$$\mathsf{ID}P = \mathsf{DI}P.$$

We conduct our proof in two steps; the first showing the points $(A_1)'$, $(B_1)', \cdots$, the corners of DIP, to lie on the sides $C'D'$, $D'E', \cdots$ of DP; the second showing them to be identical with the points $(A')_1$, $(B')_1, \cdots$ which by construction as the corners of IDP, the pentagon inscribed to DP, lie on these same sides $C'D'$, $D'E' \cdots$ of DP.

The first part of the proof consists in an application of the following well known projective theorem:

If of two simple quadrilaterals the first is circumscribed and the second is inscribed to the same conic in such a manner that the corners of the second lie in the points of contact with the conic of the first, then the four diagonals of these two quadrilaterals are concurrent.

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[1] A figure illustrating the theorem is also found in Professor Kasner's article.

To show that, for example, the point $(E_1)'$ which by construction is the point of intersection of the lines A_1C_1 and B_1D_1 lies on the line $B'C'=AD$, we construct the point of intersection E^* of the lines AB and CD and consider the two quadrilaterals A, E^*, D, E and D_1, A_1, B_1, C_1. By construction these two quadrilaterals have the mutual position described in the theorem just quoted, D_1, A_1, B_1, C_1 being the points of contact with the lines AB, CD, DE, EA of the conic inscribed to the pentagon P. Therefore the four diagonals AD, EE^*, A_1C_1, B_1D_1 are concurrent, or in a wording better suited for our proof, the point of intersection $(E_1)'$ of A_1C_1 and B_1D_1 lies on the line $AD=B'C'$.

In the same manner it follows that the points $(A_1)'$, $(B_1)'$, $(C_1)'$, $(D_1)'$ lie on the lines $C'D'$, $D'E'$, $E'A'$, $A'B'$, respectively.

It remains to be proved that the points $(A_1)'$, $(B_1)'$, $\cdots$ are identical with the points $(A')_1$, $(B')_1$, $\cdots$. For this purpose we introduce two mutually inverse plane correlative transformations defined with respect to a given pentagon with the corners O_1, O_2, O_3, O_4, O_5. These correlations shall be denoted by $\Omega(O_1O_2O_3O_4O_5)$ and $\Omega^{-1}(O_1O_2O_3O_4O_5)$; they are determined as follows:

(1)

$\Omega(O_1O_2O_3O_4O_5)$	$\Omega^{-1}(O_1O_2O_3O_4O_5)$
$O_1 \rightarrow O_3O_4$	$O_1O_2 \rightarrow O_4$
$O_2 \rightarrow O_4O_5$	$O_2O_3 \rightarrow O_5$
$O_3 \rightarrow O_5O_1$	$O_3O_4 \rightarrow O_1$
$O_4 \rightarrow O_1O_2$	$O_4O_5 \rightarrow O_2$
$O_5 \rightarrow O_2O_3$	$O_5O_1 \rightarrow O_3$.*

The proof that the corners $(A_1)'$, $(B_1)'$, $\cdots$ of $\mathsf{DI}P$ and the corners $(A')_1$, $(B')_1$, $\cdots$ of $\mathsf{ID}P$ coincide consists in various applications of the two correlations.

It is easily found from (1) that if we successively apply Ω and Ω^{-1} in the following manner: $\Omega^{-1}[\Omega(ACEBD)]$, the resulting projectivity carries the original pentagon P: A, B, $\cdots$ into the diagonal pentagon $\mathsf{D}P$: A', B', $\cdots$; so that using the symbol $\sim$ to denote projective and later also correlative relationship,

$$P \sim \mathsf{D}P.$$

The projectivity just constructed necessarily transforms the inscribed pentagon $\mathsf{I}P$ of P: A_1, B_1, $\cdots$ into the inscribed pentagon $\mathsf{ID}P$ of $\mathsf{D}P$: $(A')_1$, $(B')_1$, $\cdots$; therefore

$$\mathsf{I}P \sim \mathsf{ID}P.$$

* The fact that the five points O_1, O_2, $\cdots$ and the five lines O_2O_4, O_4O_5, $\cdots$ are really elements corresponding to each other by correlation is recognised by joining any one of the points, say O_1, to the remaining ones. The figure shows immediately that the anharmonic ratio of the four lines O_1O_2, O_1O_3 O_1O_4, O_1O_5 is equal to the anharmonic ratio of the four points in which the line O_3O_4 is intersected by the lines O_4O_5, O_5O_1, O_1O_2, O_2O_3.

In the same way as above, by constructing $\Omega^{-1}[\Omega(A_1C_1E_1B_1D_1)]$, we find

$$\mathsf{I}P \sim \mathsf{DI}P.$$

So we obtain

$$\mathsf{ID}P \sim \mathsf{DI}P \tag{2}$$

whereby, as we proved above, corresponding corners of $\mathsf{ID}P$ and $\mathsf{DI}P$ lie on the same sides of $\mathsf{D}P$[$(A')_1$ and $(A_1)'$ on $C'D'$ and so on].

It is well known that a conic uniquely determines a polarity in which a point of the conic and the tangent at that point are corresponding elements. If for the determining conic we take the one inscribed to the pentagon $\mathsf{D}P$ it follows that the corners of $\mathsf{ID}P$ and the sides of $\mathsf{D}P$ which they respectively lie on are elements corresponding by correlation. In this correlation we may replace $\mathsf{ID}P$ by $\mathsf{DI}P$ by virtue of (2). Writing the two relations thus obtained out in detail, we finally have

$$\left.\begin{array}{ll} (A')_1,(B')_1,(C')_1,(D')_1,(E')_1 \sim & (3) \\ (A_1)',(B_1)',(C_1)',(D_1)',(E_1)' \sim & (4) \end{array}\right\} C'D', D'E', E'A', A'B', B'C'.$$

The correlation defined by (3) has for its point curve of coincidences the conic tangent to the lines $C'D'$, $D'E'$, $\cdots$ in the points $(A')_1$, $(B')_1$, $\cdots$. The correlation defined by (4) has for its point curve of coincidences the conic tangent to the same lines in the points $(A_1)'$, $(B_1)'$, $\cdots$. Since however, there is only one conic tangent to five lines and since by construction $(A')_1$, $(B')_1$, $\cdots$ are the points in which the conic inscribed to $\mathsf{D}P$ is tangent to the sides $C'D'$, $D'E'$, $\cdots$ of $\mathsf{D}P$, it follows that the points $(A_1)'$, $(B_1)'$, $\cdots$ are identical with the points $(A')_1$, $(B')_1$, $\cdots$ and our proof is complete.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A SUGGESTION FOR SIMPLIFICATION OF ANNUITY FORMULAS

By W. L. HART, University of Minnesota

In any treatment of the theory of annuities certain, a fundamental problem is the preparation of formulas which will apply to the computation of the pres-

ent value and the amount of any annuity, under any arrangement concerning the compounding of interest. In the following note there is presented a single pair of simple formulas well arranged with respect to computation by means of tables, which will apply to all cases which could arise. The essence of the method consists of the recognition of the generality of the familiar formula (restricting the discussion to *amounts* of annuities)

$$S = R\frac{i}{j_p}(s_{\overline{n}|} \text{ at } i), \tag{1}$$

when it is interpreted with respect to the *interest period* rather than the *year* as the fundamental unit of time, and when p is *not* required to be an integer.[1]

Consider an investor whose interest rate per period is i, and whose interest period is specified. Let K be the periodic payment of an annuity whose term is n interest periods. Let p represent *the ratio of the interest period to the payment interval*:

$$p = \text{interest period/payment interval.} \tag{2}$$

Then, the customary methods[2] of the mathematics of investment demonstrate that the amount S of this annuity is given by the expression

$$S = K\frac{(1+i)^n - 1}{(1+i)^{1/p} - 1}. \tag{3}$$

From the generality of the data of the previous paragraph, it follows that formula 3 applies to *any* annuity under *any* arrangement as to the computation of compound interest. The simplicity of the formula results from the use of the *interest period* of the investor as the unit of time, and from the explicit introduction of the *rate per period* instead of the *nominal rate* in the final formula. Every student of the mathematics of investment has used formula 3 in instances where p is an integer. It is emphasized here that the use of this formula is equally justifiable when p is *not* an integer, because by definition p is free to assume any positive value except zero. The following illustrations exhibit the ease with which formula 3 applies to problems with data chosen as inconveniently as possible.

Illustration 1. Interest is at the nominal rate 6%, compounded monthly; the annuity consists of payments of \$100 occurring semi-annually for 5½ years.

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[1] For the details of this interpretation, for p an integer, see W. L. Hart, *The Mathematics of Investment*, D. C. Heath and Co., 1924, page 47.

[2] Hart, loc. cit., page 46.

The amount of the annuity is obtained from formula (3) with $p=1/6$, $K=\$100$, $i=.005$ and $n=66$.

Illustration 2. Interest is at the rate 4%, compounded semi-annually; the annuity consists of payments of \$5000 occurring at the end of each 6 years for 72 years. The amount of the annuity is obtained from formula (3) with $p=1/12$, $K=\$5000$, $i=.02$ and $n=144$.

The reader can easily verify that any of the usual problems in the applications of annuities would give rise to one or another of the following values of p, as defined by formula 2: $p=1, 2, 3, 4, 6, 12, 13, 26, 52, 1/2, 1/3, 1/4, 1/6$, or $1/12$. When $p=1$, formula 3 reduces to $S=K(s_{\overline{n}|}\ at\ i)$. For the other usual values of p, we adapt formula 3 for computation by means of *one* auxiliary conversion table. First let $R=pK$, and $j_p=p[(1+i)^{1/p}-1]$. Then, from formula 3 we obtain the familiar formula 1. For each desirable value of i, suppose that the values of $i\div j_p$ are tabulated for all of the usual *fractional* as well as integral values of p. By use of formula 1, the table for $i\div j_p$, together with the table for $(s_{\overline{n}|}\ at\ i)$, would serve for obtaining the amount of *any* annuity under *any* arrangement as to compound interest, except when n is not an integer, or when i is unusual. These unusual cases can be treated directly by formula 3.

It is not claimed that the method presented here is new in principle. The main object of this note was to emphasize that, with a properly prepared table for $i\div j_p$, all usual problems involving the computation of the amounts or present values of annuities can be solved by use of formula 1 and its twin (which relates to present values).

II. A Systematic Method for the Solution of Simultaneous Linear Equations

By H. G. Deming, Department of Chemistry, University of Nebraska

The method to be described is an application to linear equations of Chiò's method for the reduction of determinants.[1] The numerical work is essentially that involved in any other method. Its advantage is that it furnishes a systematic scheme of procedure, with a check against errors at every stage of the work.

We begin by replacing all the constant literal coefficients by their corresponding numerical values and transposing all terms to the left-hand side of the equality sign. The successive steps are as follows:

$$2.86x - 4.78y + 3.19z - 1.375 = 0$$

$$6.32x + 2.47y - 5.64z - 4.12 = 0$$

$$3.03x - 5.91v + 1.553z - 1.362 = 0$$

[1] See Whittaker and Robinson, *The Calculus of Observations*, London, 1924, p. 71.

(1) Tabulate the coefficients of the unknown quantities, together with the constant terms (c), as given in bold-faced type in the first few lines of the table on p. 362. The column headed s gives the actual or approximate sum of the preceding quantities in each row.

(2) We suppress one row and one column (other than column c or s) by drawing rectangles around them. In practice, *the column that is suppressed is the column* (other than c or s) *that contains the simplest numbers, preferably zeros; and the row that is suppressed is the row* (if any) *that contains a unit coefficient in the suppressed column.* In the present instance there is little room for choice. We suppress column y and row B at random.

(3) We divide each of the other numbers in the suppressed column by the number (2.47) that is found at the intersection of the suppressed column and suppressed row. The quotients, R, thus obtained, we set down in parentheses in the suppressed column.

(4) We multiply each of the quotients just obtained by each of the numbers in the other columns of the suppressed row. We place the products so obtained immediately under the numbers of the original table and pair them with the latter by braces. Thus the value of R in the first row is -1.935; and the suppressed coefficient in column c is -4.12. The product of these is $+7.973$, which is to be paired with the original number in the first row and column c.

(5) We construct a new table, using differences of the paired numbers in the preceding table. Thus the first number in the new table will be $2.86-(-12.23)=15.09$. The new table will contain one less row and one less column than the preceding table.

(6) We repeat the process, suppressing one column and one row at a time, until we finally arrive at a single row, from which the value of one unknown quantity may be found directly.

(7) We substitute this result in the last row to be suppressed, and so obtain the value of a second unknown quantity; then, substituting in preceding suppressed rows, we find the others.

(8) A continuous check is furnished by the fact that each successive value of s, when calculated from preceding values of s, will be approximately the sum of the preceding numerical quantities in the same line, so long as the work is correct. Thus, in line C, the value of s, calculated from preceding values of s, is -5.011. The actual sum of the preceding coefficients in that line is -5.010. As a final check, substitute the values of the unknowns in some line not suppressed (say A), and note that the sum of the numerical quantities so obtained is approximately zero.

	x	y	z	c	s
A	2.86 −12.23	−4.78 ($R=-1.935$)	3.19 10.913	−1.375 7.972	−0.105 1.877
B	6.32	2.47	−5.64	−4.12	−0.97
	3.03 −15.124	−5.91 ($R=-2.393$)	1.553 13.496	−1.362 9.859	−2.689 2.321
	15.09 R = 0.8312		−7.723 −9.927	−9.347 −9.327	−1.982 −4.164
C	18.154		−11.943	−11.221	−5.010
			2.204	−0.020	2.182

Thus $2.204z-0.020=0$; whence $z=0.009$. Substituting in C, $x=0.624$. Substituting in B, $y=0.092$. Check: Substituting in A,

$$(2.86\times0.624)-(4.78\times0.092)+(3.19\times0.009)-1.375=-0.001\,.$$

Our next example is one in which five unknown quantities are to be found from a set of five simultaneous linear equations. If 0 and 1 occur in a number of places in the tabulation, an opportunity is afforded for the exercise of a little skill in determining the order in which the rows and columns are to be suppressed. For it is evident that whenever a suppressed column contains a zero, all the numbers occurring in the same row with that zero may be transferred without change to the next following tabulation; and whenever a suppressed row and column intersect in a 1, we may avoid the labor of forming the preliminary quotients, R. When an entire row is exactly divisible by some constant factor, one may carry through such a division and drop out that factor; or one may multiply any entire row by any desired factor, whenever this will result in a convenience. The given equations and the tabular arrangement of the solution are as follows:

$$\begin{aligned}
x_1 \quad &+ 5x_3 - 2x_4 + x_5 - 10 = 0,\\
4x_2 - 3x_3 &+ 7x_4 \quad + 14 = 0,\\
-9x_1 + 8x_2 \quad &+ 10x_4 \quad - 31 = 0,\\
2x_1 + 20x_2 - 13x_3 \quad &+ x_5 + 2 = 0,\\
5x_1 + 4x_2 + 3x_3 + 2x_4 &+ 4x_5 - 1 = 0.
\end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	c	s
A	1	0	5	−2	1	−10	−5
	0	4	−3	7	0	14	22
	−9	8	0	10	0	−31	−22
	2	20	−13	0	1	2	12
	1	0	5	−2		−10	−5
B	5	4	3	2	4	−1	17
	4	0	20	−8		−40	−20
	0	4	−3	7		14	22
	−9	8	0	10		−31	−22
		−180	162	−18		−108	−153
C	1	20	−18	2		12	17
	1	4	−17	10		39	37
		20	−18	2		12	17
		4	−3	7		14	22
		48		−24		−81	−60
		188	−162	28		77	131
		2592		−1296		−4374	−3240
D		−16	1	8		27	20
E		−44		31		95	82
		−2404		1324		4451	3371
		R=54.636		1693.7		5190.4	4480.1
				−369.7		−739.4	−1109.1

Thus $-369.7x_4-739.4=0$; whence $x_4=-2$. Substituting in E, $x_2=\frac{3}{4}$. Substituting in D, $x_3=1$. Substituting in C, $x_1=-5$. Substituting in A, $x_5=6$. Check: Substituting in B, $0=0$.

III. A Method for Approximating Roots of Algebraic Equations in Pairs

By H. G. Deming, Department of Chemistry, University of Nebraska

The method to be described in this article consists in reducing an equation of higher degree to an equivalent set of quadratic equations, and hence gives the approximate or exact roots of the original equation in pairs. Complex or real roots may be obtained with equal ease. Moreover, it furnishes two roots with about the same labor that Horner's method requires for one.

Let the coefficient of the first term of an algebraic equation of the nth degree be unity. Detached coefficients will then appear as in the first line of the tabular arrangement below. Now let approximate values of a pair of roots be found by any of the usual methods. If the sum of the true values of these roots be represented by s, and the negative product of the true values by p, we may represent successive approximations to this sum by $s_1, s_2, s_3, \cdots$ and successive approximations to the negative product by $p_1, p_2, p_3 \cdots$.

Referring to the tabular arrangement below, the first coefficient of the original equation (unity) is multiplied by s_1, and the product, s_1, is placed in the second column. The sum of a and s_1 is then represented by a_1. Next a_1 is multiplied by s_1, and the product, a_1s_1, is placed in the third column. In this column is also placed the product of the first coefficient (unity) and p_1. Proceeding in this way, we complete the upper portion of the tabulation. Then, repeating the process, starting from the *first reduced coefficients*, $1, a_1, b_1, c_1, \cdots$, just obtained, we obtain a further set of quantities, which we shall call partials, and designate by $A, B, C, \cdots$. The final partial, N, need not be calculated.

Note that the work is carried out to the column beyond n, in the third row. This affords a check, for the sum of the quantities in the second row is then s_1S_1, and the sum of those in the third row is p_1S_1, where S_1 is the sum of those in the fourth row (excluding n_1). A similar check is applied to the last three lines of the tabulation.

	1	a	b	c	$\cdots$	l	m	n		*Sums*
s_1		s_1	a_1s_1	b_1s_1	$\cdots$	k_1s_1	l_1s_1	m_1s_1		s_1S_1
p_1			p_1	a_1p_1	$\cdots$	j_1p_1	k_1p_1	l_1p_1	m_1p_1	p_1S_1
	1	a_1	b_1	c_1	$\cdots$	l_1	m_1	(n_1)		S_1
s_1		s_1	As_1	Bs_1	$\cdots$	Ks_1	Ls_1			s_1S
p_1			p_1	Ap_1	$\cdots$	Jp_1	Kp_1	Lp_1		p_1S
	1	A	B	C	$\cdots$	L	(M)			S

Having completed and checked the tabulation, the preliminary values, s_1 and p_1, may be improved by adding the corrections,

$$(1) \qquad dp_1 = \frac{Mm_1 - Ln_1}{L^2 - KM}, \quad ds_1 = \frac{m_1 + Kdp_1}{-L}.$$

These corrections are best worked out roughly with a slide rule. Then, with the corrected values, $s_2 = s_1 + ds_1$, and $p_2 = p_1 + dp_1$, we may go back to the detached coefficients of the original equation and operate on these again, to obtain a set of *second reduced coefficients*, $1, a_1, b_2, c_2, \cdots$. New values for the partials need not be calculated. A second correction for the preliminary values of s and p may then be found by replacing m_1 and n_1, in the preceding equations, by m_2 and n_2. Repeating this process indefinitely, say r times, we may obtain the sum and product of the roots with any required degree of precision.

Finally, each of the reduced coefficients must be corrected by adding small quantities, which are obtained as in the following scheme:

Partials:	1	A	B	C	$\cdots$	J	K	L	M	
Reduced coefficients:	1	a_r	b_r	c_r	$\cdots$	j_r	k_r	l_r	m_r	n_r
Corrections:		ds_r	Ads_r	Bds_r	$\cdots$	Ids_r	Jds_r	Kds_r	Lds_r	Mds_r
			dp_r	Adp_r	$\cdots$	Hdp_r	Idp_r	Jdp_r	Kdp_r	Ldp_r

The reduced coefficients, thus corrected, are the coefficients of an equation of lower degree, whose roots are those of the original equation, excluding the two roots whose sum and product have just been found.

PROOF: The coefficients of the original equation are the sums of all the products that may be formed from its roots (with signs reversed); these roots being taken one at a time in forming a, two at a time in forming b, and so forth. The process outlined, leading to the reduced coefficients, consists in subtracting from these sums all the root-products that contain either one or both the two given roots as factors. It therefore leads to the coefficients of an equation having all the roots of the original equation save the two given roots.

Now if corrections, ds and dp, are added to the preliminary values of s and p, the corrections that accumulate from column to column by the process outlined may be shown to follow the law that is indicated in the second preceding paragraph. The partials, $1, A, B, C, \cdots$ are in fact the *partial derivatives* of the coefficients with respect to s and p separately. But the reduced coefficients, m_1 and n_1, in the two final columns represent errors that have accumulated in these columns because of the use of inexact values of s and p. These are to be reduced to zero, as nearly as may be, in the next operation. Accordingly,

$$m_1 + Kdp_1 + Lds_1 = 0, \qquad n_1 + Ldp_1 + Mds_1 = 0.$$

From these we may obtain the equations that have been given for calculating ds_1 and dp_1. The partials are given exactly by the process we have described

only when s and p are known exactly from the beginning. But in practice even a very rough calculation of their value will suffice, since they are employed only in finding small corrections to be applied to the coefficients.

Example: As a numerical example consider the equation

$$x^6+7x^5-53x^4-10x^3-180x^2-470x-150=0\,.$$

A rough preliminary survey, plotting a small portion of the curve in the neighborhood of the origin, will reveal four approximate roots, -11, -1.3, -0.4, $+5$. But the sum of all the real and complex roots of the equation is the negative of the second coefficient, namely, -7. Comparing this with the sum of the four known roots, we see that another pair of roots must exist, whose sum is approximately $+1$. Again, the product of the real and complex roots of the equation, with signs reversed, is the final coefficient, -150. Comparing this with the product of the four known roots, we see that the product of the missing pair of roots must be about $+5$. Accordingly, we detach coefficients and proceed as follows:

	1	7	− 53	− 10	− 180	− 470	− 150
$s_1 = 1$		1	+ 8	− 50	− 100	− 30	0
$p_1 = -5$			− 5	− 40	+ 250	+ 500	+ 150
	1	8	− 50	− 100	− 30	0	0

Since the final coefficients have both become zero, our assumed values of s_1 and p_1 were exact, hence the original equation has a pair of (complex) roots which are the roots of the quadratic, $x^2-x+5=0$. If the original equation is divided through by this quadratic, we obtain a quartic equation, whose coefficients are those appearing in the final line of the outline above and whose roots are the real roots of the original equation. Let us determine these to four decimal places.

We may begin with the pair of roots whose approximate values have been given above as -1.3 and -0.4. Their sum is $s_1=-1.7$, and their negative product, rounded, is $p_1=-0.5$.

	1	8	− 50	− 100	− 30		*Sums*
$s_1 = -1.7$		− 1.7	− 10.71	+ 104.057	− 1.542		90.105
$p_1 = -0.5$			− 0.5	− 3.15	+ 30.605	− 0.453	26.502
	1	6.3	− 61.21	$m_1 = 0.907$	$n_1 = -0.937$		− 53.003
$s_1 = -1.7$		− 1.7	− 7.82	+ 118.20			+ 108.68
$p_1 = -0.5$			− 0.5	− 2.3	+ 34.765		+ 31.965
	1	$K = 4.6$	$L = -69.53$	$(M = 116.8)$			

$Mm_1 = 105.93$ $Ln_1 = 65.15$ $L^2 - K\,M = 4300$ (*approximately*).

Thus, from equations (1),

$$dp_1 = +0.0095, \quad ds_1 = +0.014.$$

With the corrected values, $s_2 = s_1 + ds_1 = -1.686$ and $p_2 = p_1 + dp_1 = -0.4905$, we repeat the first part of the preceding work. This gives,

$$m_2 = +0.0220. \qquad n_2 = -0.0499$$

$$dp_2 = -0.00021 \qquad ds_2 = +0.0003$$

$$p_3 = p_2 + dp_2 = -0.4907 \qquad s_3 = s_2 + ds_2 = -1.6857.$$

The two roots sought are accordingly those of the quadratic

$$x^2 + 1.6857x + 0.4907 = 0,$$

namely -1.3115 and -0.3742.

The reduced coefficients obtained in the last operation are corrected as follows:

Partials:	1	4.6	− 6 .53	+ 116.8	− 0.04
Uncorrected coefficients:	1	6.314	− 61.1359	+ 0.0220	− 0.0499
Corrections: $ds_2 = 0.0003$		+ 0.0003	+ 0.0014	− 0.0209	+ 0.0350
$dp_2 = -0.00021$			− 0.0002	− 0.0009	+ 0.0146
Corrected coefficients:	1	6.3143	− 61.1347	+ 0.0002	− 0.0003

A check is afforded by the fact that the two final coefficients now practically vanish. The remaining roots are evidently those of the quadratic

$$x^2 + 6.3143x - 61.1347 = 0,$$

namely —11.5894 and +5.2750.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Algebraic Arithmetic. By ERIC T. BELL. American Mathematical Society, Colloquium Publications, Volume VII. New York, 1927. iv+180 pages.

Between the classic arithmetic on the one hand, as developed by the school of Gauss, and the modern analytic theory of numbers on the other hand, in

which limiting operations usually appear both in the arguments and in the conclusions, lies an extensive intermediate region of the theory of numbers "where the methods of algebra and analysis are freely used to yield relations between integers expressed wholly in finite terms and without reference, in the final propositions, to the operations or concepts of limiting processes." To this part of the theory of numbers Bell has given the name *algebraic arithmetic*. It is not desirable that its boundaries should be sharply defined. In it may be placed many results in the theory of numbers which do not belong specifically either to the classical theory or to the modern analytic theory. During recent times, and especially in the hands of Bell himself, this part of the theory of numbers has had a rapid development.

It is the purpose of this book "to outline a few promising directions in which progress may be made toward classifying, extending, and generalizing the methods and results of algebraic arithmetic. The insistence will be upon general methods rather than specific applications, as the latter are so numerous, and so readily made from the general formulations, that it will be sufficient merely to indicate occasionally a few of them to lend concreteness to the abstract theories. What is given here is but a narrow cross section of a very extensive field." As is indicated by this quotation from the first page, this book is devoted almost entirely to a general and abstract formulation of the problems and the methods of this relatively new division of the theory of numbers.

The character of the work may be partially indicated very simply by means of a special but a typical case of one of the principal concepts on which the methods are based, namely, the concept of paraphrases. Suppose that we have obtained in some way, say from the theory of elliptic or theta functions, a linear identity among the sines of the angles $a_i x + b_i y$ for the set 1, 2, 3, ... of values of i. Let $f(a, b)$ be an arbitrary odd function of its two arguments. Then Bell shows that we have a like linear identity among the $f(a_i, b_i)$. Since the function f is at our disposal we may obtain as many arithmetical facts as we please by specializing this function in various ways. From similar identities among cosines other arithmetical facts may be similarly obtained by means of even functions. These results are extended to functions of various types of mixed parity. All this is but one particular type of application of a rich and extended theory set forth here for the first time in its general and abstract form.

The book contains such a wealth of material and in such an abstract form that one can come to a full appreciation of its scope only by an extended study, supplemented by the development of many of its numerous applications. It is safe to predict that there will be a large development of algebraic arithmetic taking its rise from this book and from the researches of Bell on which it is based.

R. D. CARMICHAEL

Mathematical Preparation for Physical Chemistry. By FARRINGTON DANIELS. McGraw-Hill Book Co., 1928. x+308 pages.

This surprisingly interesting book was written by a professional chemist and it is extremely gratifying to the mathematician to find an American chemist who knows so much mathematics.

As a source of problems the teacher or the writer of undergraduate mathematics text-books will find the book of value. Problems from chemistry afford an interesting variation from those more usually given. The reviewer was particularly impressed with the common sense approach to many topics and with the discussion of various simple things frequently omitted from ordinary courses. The book is decidedly an addition to text-book literature. The author says that it was based on a one semester course for sophomores, but he is not proposing to substitute it for the usual courses. In fact it is hard to believe that the necessary time should be taken in the sophomore year for such a course. It seems that it could be given later as a review to students of chemistry who had had the usual courses but who wished to freshen their memories on their mathematics and to study its applications within their special field. Such a book should also be of service to the mature chemist who wishes to study mathematics without a teacher in order to read modern literature on physical chemistry.

The contents is partly indicated by the chapter headings: Large and Small Numbers; Logarithms; The Slide Rule; Graphical Representation of Equations; Graphs of Equations of the Second Degree; Graphs of Logarithmic and Trigonometric Functions; Differential Calculus; Differentiation; Graphs and Calculus; The Differential; Integral Calculus; The Significance of "e"; Differentiation and Integration of Trigonometrical Functions; Integration; The Use of Integration Tables; Geometrical Applications of Integral Calculus; Partial Differentiation; Differential Equations; Infinite Series; Probability; Graphical Methods in Physical Chemistry. There is a variety of tables. Such a book is clearly mathematics and not chemistry. It is in the problems and in some discussions that chemistry particularly enters.

And now to the task of calling attention to things about which a mathematician can not be enthusiastic. These consist primarily, although not wholly, in the use of some terms that are in bad repute with mathematicians and in an undue laxness in the treatment of limits. At times the reviewer is in doubt if the author himself really always thinks of his calculus as a study of certain limits or if in his thinking he is retaining remnants of the abandoned "little zero" point of view. As we all know there is a very real difficulty in being rigorous and at the same time understandable to a sophomore. In fact it is likely that no one of us objects to some frank and admitted lapses from rigor in undergraduate teaching. In the opinion of the reviewer, however, the author goes too far in this direction, particularly when discussing the foundations of

calculus. Extensive quotations might be given; but because of limitations of space reference is made to the book itself and there to but four places: viz., p. 86; first three lines of p. 78; first paragraph of p. 114; lines 7 to 9, p. 115. The reviewer feels that various definitions are too broad, not conforming to common usage, as for example that of calculus (p. 70), and that some are inadequate, as that of a determinant (p. 260). Some terms with apparent uses to which the reviewer takes exception are: "finite numbers" to distinguish certain numbers from differentials; "quantities" as meaning *numbers*; "limiting value" as meaning *limit*; "becomes" sometimes as meaning *is* and sometimes as meaning *approaches*; "equation" as meaning *function*; "drops out" as meaning *approaches zero*; "order of an equation" as meaning *degree of an equation*; "series" as meaning *sequence*; "approaches infinity"; and a quite too frequent use of "evident."

Not many typographical errors were noted.

TOMLINSON FORT

Freshman Mathematics. By G. W. MULLINS and D. E. SMITH. Boston, Ginn and Company, 1927. vi+386 pages.

The purpose of this text is to present a fairly comprehensive survey of the mathematics of first year college work through an approach by way of "the common uses of the science." It is in this sense then an introduction to some of the topics of college algebra, trigonometry, analytic geometry, and calculus which are most immediately applicable to practical affairs. It is designed as a one year course for students who have not necessarily had more than an "elementary course in algebra such as is offered in every high school, a familiarity with the important basal propositions of plane geometry, and a knowledge of the simple rules or formulas of mensuration commonly given in arithmetic or in intuitive geometry." Consequently it does not wholly meet the needs of one who has already received considerable training in the formal developments of algebra, trigonometry, and geometry except by a selection of topics and supplementary material.

In place of the "conventional formal review of algebra" the book has an opening chapter on applications of elementary algebra through a study of the formula, the equation, graphs, and exponents. The binomial theorem, progressions, and logarithms follow in two more chapters. The next chapter, which is on trigonometry, is excellent as far as it goes, beginning with a study of general angles and introducing the trigonometric ratios as functions of the general angle. With practical applications in view, everything leads up to the solution of triangles. Short four place tables are given at the end of the text, but it seems advisable to use a more extended separate set of five place tables with proportional parts in order to develop speed of computation. Analytic trigonometry must be slighted next with one page of discussion comprising very

scant mention of trigonometric identities and equations. All the rich field of fundamental identities, addition formulas, multiple and half angle formulas and inverse functions receive no treatment.

A chapter on analytic geometry with eight sub-headings follows. The increment notation of the calculus is early introduced; the conics and a few miscellaneous loci are studied. A great many things must of course be omitted, such as transformation of coördinates, equations of curves from empirical data, and polar coördinates, which surely must have been overlooked. Pages 239–316 take up the calculus applied to algebraic functions only. Application of the calculus is made to rates, and to maximum and minimum problems. Integration as the inverse of differentiation is considered and applied to finding the area under a curve by use of "rate of change of area." A short chapter on numerical equations, giving approximate graphic solution and a good discussion of Newton's method completes the more usual work. The last chapter is made up of a few basic formulas in practical mensuration in the plane and space, followed by lists of exercises involving the use of these formulas.

Emphasis throughout the text is placed upon practical problems, with exercises in which the results are not simple integers, and with equations and formulas with decimal coefficients. Historical notes, which were found to stimulate interest, are given at appropriate places. The explanation and discussion is quite detailed and clear, making it easily understood by a student who has had little rigorous mathematical training.

The reviewer has had experience with several texts in so-called "unified" or "combined" courses in which the texts were variously designed, and he feels that this text has very well met the purposes for which it was intended. It seems an admirable text for use in the agricultural, biological, and liberal courses for students who want a little working knowledge of elementary mathematics including the calculus but who do not intend to go much farther. The authors of *Freshman Mathematics* certainly have accomplished this aim, for the text has proved a success in courses such as those mentioned above.

The book is well printed with clear type and well drawn figures. No answers are given; thus the student is compelled to depend upon his own accuracy. More emphasis might have been placed upon checks and check formulas; but this might well be left for the instructor to develop as he must have some excuse for his presence at the class meeting.

CLYDE M. HUBER

Exercices de Géométrie Moderne. Vol. I, Géométrie dirigée, 132 pp., 13 francs. Vol. II, Transversales, 81 pp., 9 fr. Vol. III, Division et faisceau harmoniques, 86 pp., 9 fr. Vol. IV, Pôles, polaires, plans polaires, dans le cercle et la sphère, 125 pp., 13 fr. Vol. V, Rapport anharmonique, 101 pp., 11 fr. Vol.

VI, Inversion, 134 pp., 14 fr. Vol. VII, Homographie, 128 pp., 14 fr. Vol. VIII, Involution, 93 pp., 11 fr. Vol. IX, Géométrie projective. Application aux coniques, 172 pp., 15 fr. In-8. By G. PAPELIER. Paris, Vuibert, 1925–1927.

Each of these little volumes is written according to the same plan. The author establishes first the fundamental theorems relating to the subject at hand, and then proceeds to apply them to the solution of various problems. As one might expect, the line of demarcation between theory and application is not always very definite. The theory of orthogonal circles, for instance, and the general theory of coaxal circles, for that matter, are incorporated in the applications of Vol. III. Then again the choice of the problems solved as applications is of necessity a matter of personal preference, perhaps to some extent even a matter of mere chance. The author's selections are as good as any can be, and most of his examples have also a good deal of intrinsic interest.

The treatment is almost exclusively synthetic and presupposes, on the part of the reader, nothing beyond the rudiments of analytic geometry. The proofs and solutions are presented fully and explicitly, but without undue length. The book will interest any college student with a taste for geometry and will be stimulating to anyone engaged in teaching secondary mathematics. It is regrettable that the reader is offered no opportunity to exercise his own sagacity, as there are no unsolved problems in any of these volumes. Long lists of such exercises can be collected with little trouble. And while expressing wishes, may we be allowed to say that it would be desirable to have a similar series of volumes dealing with the geometry of the triangle, and, perhaps, the tetrahedron as well.

NATHAN ALTSHILLER-COURT

The Mathematics of Engineering. By RALPH E. ROOT. The Williams and Wilkins Company. xiii+540 pages. $7.50.

On opening this book the reader is struck with the clear but unconventional type used by the printers. As he reads the book he is struck with the clear but unconventional proofs used by the author. It is amazing how much mathematics is dealt with in this book (calculus, analytics, parts of algebra, differential equations, some complex functions, empirical data, etc.). The book is almost a cours d'analyse. After reading in the preface and the table of contents about the amount of mathematics that is to be discussed in the following pages, you find yourself in the midst of a leisurely and surprisingly full treatment of functions, of limits and of continuity, and you wonder how at this rate the author will ever cover the ground. However, you find that the discussions later on in the book are not so complete; also that the author (during the thirteen years in which he has gradually accumulated the material of the text) has re-

vised his proofs until many of them are really marvels of brevity, ingenuity, and originality. The book is very stimulating and refreshing, the various subjects are handled vigorously and fairly rigorously. The whole text uses mechanical, geometrical, and dynamical ideas that ought to appeal to engineers. It is intended as a second course in analytics and the calculus that will lead up to more advanced topics in mathematics. It seems as though engineers would welcome such a book in their libraries for frequent reading and reference. In spite of all the topics that the careful and complete index shows that the book touches on, the text does not seem cluttered up or crowded, mostly because of the completeness of some of the sections.

Before going on with the good points of the book, we notice that it has entirely too many typographical errors, some of them very serious ones. Also in some places where fullness of discussion would seem to be desirable the author is too brief, almost descending to a handbook-like statement of methods. Thus he is unsatisfactory when he treats of areas on page 222; and his interpretation of successive integration on page 244 does not clarify the hazy ideas gained in an elementary course in the calculus. It seems as though here the engineer should be shown why he can take the different kinds of elements of area, volume, etc. Also would it not be better to *define* volume as the limit of a sum of infinitesimal volumes, moment of inertia as the limit of a sum of infinitesimal moments of inertia, etc., instead of saying that the volume of a surface *can be obtained* as the limit of a sum of volumes and the moment of inertia *can be obtained* as the limit of a sum of moments. On page 82 he speaks of the integral as the limit of a sum of *differentials*, whereas using the ordinary definitions of differential and of infinitesimal we should call the integral the limit of a sum of *infinitesimals*. He does not justify the method of taking the derivatives of implicit functions. On page 371 he speaks of the area of a surface and shows how to obtain it by integration, whereas logically we define the area of a surface as the limit of a sum of infinitesimals and show how this leads up to the integral. On page 170 and 185 the author is not clear as to what he means by "being at infinity." When discussing Taylor's expansion, it would have been well to show under what circumstances one can expand a function in such a series. The book might contain a better discussion of change of variable and of limits in integration, giving the logic of the situation, the conditions that must be satisfied, etc. No mention is made of the discriminant of a conic, although otherwise the testing and reducing of nondegenerate conics is well done.

The fact that the book has so very many good points makes one notice with surprise the occasions chronicled above where "Homer nods." It would be difficult even to catalogue the interesting points in the text. When discussing the graphs of curves the author points out carefully that in the applications of such functions as $y = \cos x$ the variable x very seldom represents an angle, but

may be time, temperature, length, etc.; also he notes that only certain parts of curves are useful in applications. How often do professors of engineering complain that their students do not learn the above facts in mathematics! On page 32 he gives such problems as "suggest several functions $f(x)$ for which $\lim_{x\to 0+} f(x) = +\infty$." Similarly in Ex. 3 on page 27 we find "show that $\lim_{x\to 0} \sin x = 0$. (Show the way in which the choice of δ depends on the given ϵ)." Would that more such searching examples were given in text books, where the student has to invent illustrations of theorems or has to make up problems and then solve them. On page 43 we find a definition of the derivative that must delight the heart of an engineer, namely "The derivative of a function of x with respect to x is the instantaneous rate of change of the value of the function measured in terms of the rate of change of x at the same instant."

The figures throughout the book are excellent and are very cleverly used in the proofs of formulas. For example on page 54 $(d/dx) \sin x = \cos x$ is proved very deftly from a figure, also for $(d/dx)e^x = e^x$ a figure is used (on page 59). Moreover $(d/dx)e^x$ is discussed before $(d/dx) \log x$. This is certainly an innovation. The proof depends on considering all the curves $y = \alpha^x$ with α having different values, then choosing the value of α for which the corresponding curve has unit slope for $x = 0$. The only criticism one might make is that it is somewhat of an assumption to say that there exists such a curve $y = \alpha^x$ with unit slope for $x = 0$. On page 164 the formulas for rotation of axes follow beautifully from a figure. Good graphs of curves are drawn for the discussion of derivatives, of maxima and minima, of points of inflection, of integrals (areas, etc.). But the most astonishing of all is the way he treats conics on page 169, defining them from a figure as sections of a cone, using this to get their definition in terms of foci and directrices, and finally obtaining the simpler forms for their equations.

The scope of the book is well brought out by noting that on page 98 we find a clear discussion of determinants (later on comes Sylvester's "dialytic method" for the elimination of a variable between two equations). Families of curves are treated; asymptotes of hyperbolas are neatly discussed on page 176; the discriminant of a cubic equation in one variable is derived; on page 261 is a fine discussion of indeterminant forms; on page 268 Taylor's theorem is derived by integration; on page 282 we find a brief proof of the continuity of $\sum u_i(x)$ by means of a "majoring series;" on page 291 the function $e^{i\theta}$ is introduced; on page 298 we are set to work mapping complex functions; on page 330 there is a short treatment of lines in space; etc. The problems are very well chosen so as to bring out applications of the theory, to drive home the topics treated, and to lead on to more advanced subjects. For example, on page 210 we find the problem of the best angle at which to elevate a gun in order to get the greatest possible range; while under derivatives we find problems that are really introductory to differential equations. Moreover the variables used throughout the book are such

as fit the problems, such as t for time, s for space. In fact there is nothing pedantic anywhere in the text. At the end of many chapters are lists of references for further study.

Most of the definitions are very good, for instance that of the length of arc (on page 217). The author has such a thorough and virile mastery of his subject that he is able to choose (or invent) the briefest possible discussions, and those best adapted to his purposes. Thus on page 211 he turns the usual procedure right around and first derives the equation of the osculating circle of a curve at a point; then he gets its radius and center. Mathematicians very often fall into the rut of sticking to so-called "classical" definitions and methods of discussion, sometimes one writer of a textbook slavishly copying an earlier one. To read such a unique, hearty book as this one is as refreshing as a draught of cold water to a thirsty traveler.

ALAN D. CAMPBELL

Elementary Solid Geometry. By V. B. NAIK AND S. B. BONDALE. Aryabhushan Press, Poona, India, 1926. vi+220 pages.

This is the work by two professors of mathematics in India, one in Fergusson College and the other in Willingdon College. It covers the course prescribed by the Bombay University for intermediate examinations, and adds a number of other topics that are not required for this purpose. These include the mensuration and certain properties of the sphere, rectangular coordinates, perspective, direction cosines, the tetrahedron and its inscribed and circumscribed spheres, Euler's theorem, regular solids, and the frustum of a cone. It contains certain features that will appeal to teachers of the subject in the New World, as in the case of its simple introduction, based upon commonplace experiments which lead to a feature wholly lacking in Euclid and not sufficiently in evidence in our progressive texts of today; namely, a satisfactory working set of postulates. These assumptions are classified as (1) General Postulates, (2) Straight-Line Postulates, (3) Plane-Surface Postulates, (4) Parallel-Lines Postulates, and (5) a Two-Planes-in-Space Postulate. Interspersed among these assumptions are the definitions needed in the course.

In the preliminary work the authors have shown considerable ingenuity in selecting the simple experiments mentioned. This is seen in such features as the tests to be applied to a surface to determine if it is a plane and to a stretched thread to see if it is approximately straight; to trying to fold and crease a piece of paper on uneven ground; and to explaining why the shadow of the top of a vertical wall is parallel to the foot of the wall.

The proofs are less formally arranged on the page than in American texts, following rather the method of most editions of Euclid. The basic propositions are of course much like those in this country, but they are fewer in number,

the pupil's interests being directed to numerous deductions from a few standard theorems rather than to a few deductions from a larger number. As already stated, the authors extend the work to include several limited explorations of territory not investigated in our textbooks. For this reason and because of the originality shown in many of the deductions, the book is well worth reading.

It can not be expected that typographically or graphically the work should measure up to what is done in Europe or America, but it is a praiseworthy attempt to give, under unfavorable circumstances, not only a clear page but a set of figures that elucidate adequately the text.

American teachers of the present time would object to such a massing of definitions and postulates as is here found, although this occurs after the pupil has been prepared to understand them, so that the objection is not serious. Probably the chief objection would, however, be that parts of the work are too difficult for the American pupil, although required of the schoolboy or in the early college years on the Indus and the Ganges. If this be the case, we should not be particularly proud of our criticism. High-school teachers or instructors in freshman classes who are interested in the subject might try their pupils upon a few such theorems and exercises as these:

The normal to a surface of revolution at any point in it intersects or is parallel to the axis.

Two surfaces which touch one another at a point have a common normal at that point.

A sphere rests on a hole of the shape of an equilateral triangle of side 5 in. If the radius of the sphere is 3 in., what is the maximum height of the sphere above the plane of the hole?

In a regular tetrahedron in which each dihedral angle is β,

$$\beta = \sin^{-1}(2\sqrt{2})/3 = \cos^{-1}(1/3) = \tan^{-1}2\sqrt{2}.$$

If OP makes angles x, y, z with the axes of coordinates, $\cos^2 x+\cos^2 y+\cos^2 z=1$.

If two figures in different planes are in perspective, corresponding straight lines either meet in collinear points or are parallel.

Of course these statements are mathematically of no difficulty, but our high-school pupils would hardly care to encounter them on a college entrance examination for honors in mathematics, and our college students would be surprised if they met with them in their courses. Perhaps we can take comfort in thinking that a Hindu boy would not know what to do with an American multiple-response test; but possibly this thought would have the opposite effect, though we would not confess it.

DAVID EUGENE SMITH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3332. Proposed by R. E. Gaines, University of Richmond.

In a given ellipse the two conjugate diameters are drawn which are equally inclined to the major axis, and a similar variable ellipse touches these two diameters and has its major axis on the same line as that of the fixed ellipse. Find the distance between their centers when the area common to the two ellipses is a maximum.

3333. Proposed by W. H. Roever, Washington University.

Consider the system of two partial differential equations

$$X_i(f) \equiv a_{1i}\frac{\partial f}{\partial x_1} + a_{2i}\frac{\partial f}{\partial x_2} + a_{3i}\frac{\partial f}{\partial x_3} = 0, \quad i = 1, 2,$$

in which the coefficients a_{ji} are functions of x_1, x_2, x_3. From one point of view, a necessary and sufficient condition that these equations have a common integral surface is

$$\Theta \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ X_1(a_{12}) - X_2(a_{11}) & X_1(a_{22}) - X_2(a_{21}) & X_1(a_{32}) - X_2(a_{31}) \end{vmatrix} \equiv 0.$$

and from another point of view, such a condition is

$$\Omega \equiv U\left(\frac{\partial V}{\partial x_3} - \frac{\partial W}{\partial x_2}\right) + V\left(\frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3}\right) + W\left(\frac{\partial U}{\partial x_2} - \frac{\partial V}{\partial x_1}\right) \equiv 0,$$

in which

$$U = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}, \quad V = \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix}, \quad W = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}.$$

Show that $\Theta \equiv \Omega$.

3334. Proposed by James Singer, Graduate College, Princeton, N. J.

Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{2^n}\left(\sum \frac{e_1}{2^{2\cdot 1-e_1}} + \frac{e_2}{2^{2\cdot 2-(e_1+e_2)}} + \cdots + \frac{e_n}{2^{2n-(e_1+e_2+\cdots+e_n)}}\right)\right],$$

where $e_i = 0$ or 1, and the sum is extended over all possible choices of the e's; i.e., the sum of the n terms where all the e's except one are zero, plus the $n(n-1)/2$ terms where all the e's except two are zero, $\cdots$, plus the single term where all the e's are unity.

3335. Proposed by Paul Wernicke, Washington, D. C.

Assume, on the sides of a triangle ABC, the points L on BC, M on CA, N on AB. Find the condition for the existence of a real straight line dividing the three joins AL, BM, CN in the same ratio, and, in particular, bisecting them.

3336. Proposed by Otto J. Ramler, The Catholic University of America.

Construct a triangle having given $a-b$, h_b+h_c, and the angle A, where a and b are sides, and h_b, h_c are the altitudes upon the sides b, c, respectively. (Altshiller-Court's College Geometry, p. 28, no. 7).

3337. Proposed by E. B. Escott, Oak Park, Illinois.

Sum the series,

$$\frac{6^3}{2.3.4.5}x^5 - \frac{8^3}{4.5.6.7}x^7 + \frac{10^3}{6.7.8.9}x^9 - \cdots$$

3338. Proposed by N. A. Court, University of Oklahoma.

The two pairs of extremities of two harmonic segments determine an involution in which the midpoints of the segments are two conjugate points.

3339. Proposed by Paul Capron, U. S. Naval Academy.

Discuss the nodes of the parabolic spiral $(\rho-a)^2=ca\theta$.

3340. Proposed by V. M. Spunar, Chicago, Illinois.

Solve

$$\begin{vmatrix} x^2dx, & y^2dy, & z^2dz \\ dx, & dy, & dz \\ x, & y, & z \end{vmatrix} = 0$$

consistent with the equation $xyz=1$.

SOLUTIONS

238 [1916, 19]. Proposed by C. N. Mills, Brookings, South Dakota.

Determine the rational values of x that will render x^3+px^2+qx+r a perfect cube. Apply the result to $x^3-8x^2+12x-6$.

Partial Solution by R. E. Moritz, University of Washington

Since the given expression may be written $(x+p/3)^3+(q-p^2/3)x+(r-p^3/27)$, a rational value of x which renders it a perfect cube is given by $x=(27r-p^3)/9(p^2-3q)$, where it is assumed that p, q, r are rational numbers. In the given numerical example $x=25/18$ gives for the expression the value $(-23/18)^3$. This is not the only solution, for $x=1$, -1 give the values $(-1)^3$, $(-3)^3$.

470 [1915, 228; 1919, 414]. Proposed by R. E. Moritz, University of Washington.

Prove that

$$\theta = (\lambda + (q/p)\mu)\pi, \quad (\lambda = 1, 2, 3, \cdots, q-1;\ \mu = 0, 1, 2, \cdots, p-1),$$

and

$$\theta = (2\lambda - 1)\pi/2 + (q/p)(4\mu \pm 1)\pi/2, \qquad (\lambda = 1, 2, 3, \cdots, (q-1)/2; \quad \mu = 0, 1, \cdots, p-1),$$

determine the same set of points on the curve $\rho=a\cos(p/q)\theta$, where p and q are two odd integers without a common factor, and a is any constant.

Solution by Otto Dunkel, Washington University

For the given equation two points corresponding to θ_1 and θ_2 will coincide in two cases: if $\theta_2-\theta_1$ is an odd multiple of π and $\rho_2=-\rho_1$; if $\theta_2-\theta_1$ is an even multiple of π and $\rho_2=\rho_1$. Designate by (P) the set of points given by the second formula for the angle. We may write this formula as

$$(P) \qquad \theta' = \left[(2\lambda' - 1) + \frac{q}{p}(2\mu' - 1)\right]\frac{\pi}{2}, \qquad \begin{matrix}\lambda' = 1\ 2, \cdots, (q-1)/2, \\ \mu' = 0, 1, \cdots, (2p-1).\end{matrix}$$

If in this formula we replace μ' by $\mu'-p$, the angle θ' is replaced by $\theta'-\pi q$. Also $\cos(p/q)(\theta'-\pi q) = -\cos(p/q)(\theta')$. Since p and q are odd the two points coincide. Hence the values $\mu'=p$, $(p+1)$, $\cdots$, $(2p-1)$ give the same points as the remaining values $\mu'=0, 1, \cdots, (p-1)$. We shall consider only the points of this set which we shall denote by $(P)'$.

Now make the transformation $\lambda'=\lambda''-(q-1)/2$, $\mu'=\mu''+(p+1)/2$, and we may then write

$$(P)' \qquad \theta' = \left(\lambda'' + \frac{q}{p}\mu''\right)\pi, \qquad \begin{matrix} \lambda'' = (q+1)/2, & (q+3)/2, \cdots, (q-1), \\ \mu'' = -(p+1)/2, & -(p-1)/2, \cdots, -1, 0, 1, \cdots, (p-3)/2. \end{matrix}$$

In this last formula λ'' and $q-\lambda''$ give the two angles θ_1' and θ_2' such that $\theta_2'-\theta_1'=(q-2\lambda'')\pi$ and $(p/q)\theta_2'+(p/q)\theta_1'=(2\mu''+p)\pi$. Hence these values give the same point and we may replace λ'' by $\lambda=q-\lambda''=(q-1)/2, (q-3)/2, \cdots, 1$. We shall now make a change in the sequence for μ''. If we compare the angles θ_1', θ_2' given by μ'' and $p+\mu''$, we have $\theta_2'-\theta_1'=q\pi$ and $(p/q)\theta_2'-(p/q)\theta_1'=p\pi$. Hence in the sequence for μ'' we may replace the $(p+1)/2$ negative values by $(p-1)/2, (p+1)/2, \cdots, (p-1)$, and we may write

$$(P)' \qquad \theta' = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \begin{matrix} \lambda = 1\ 2, \cdots, (q-1)/2, \\ \mu = 0, 1, \cdots, (p-1). \end{matrix}$$

Here again λ and $q-\lambda$ give the same points and we obtain the set $(P)'$ if we set $\lambda=(q-1), (q-2), \cdots, (q+1)/2$. Hence we have finally

$$(P) \qquad \theta = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \begin{matrix} \lambda = 1, 2, \cdots\ (q-1), \\ \mu = 0, 1, \cdots, (p-1). \end{matrix}$$

Also solved by A. J. CAVANAGH and the PROPOSER.

472 [1915, 267]. Proposed by Paul Capron, U. S. Naval Academy.

The sides of a spherical triangle are a, b, c; the corresponding opposite angles are A, B, C; p and P are the polar distances of the inscribed and circumscribed circles; $a+b+c=2s$; $A+B+C=2S$. From a geometric figure, by the formula for solving right spherical triangles, show that

$$\tan^2 p = \operatorname{cosec} s \sin(s-a) \sin(s-b) \sin(s-c); \tag{1}$$

$$\cot^2 P = -\sec S \cos(S-A) \cos(S-B) \cos(S-C). \tag{2}$$

Thus establish the usual formulas for the tangent of the half-sides and half-angles.

Also show that

$$\frac{\text{sine of angle}}{\text{sine of the opposite side}} = \frac{\cot P \cos S}{\tan p \sin s}. \tag{3}$$

SOLUTION BY OTTO DUNKEL, Washington University

The formula (1), the law of tangents for angles, and the formula for the tangent of the half-angle are established in the desired manner in Kenyon and Ingold's *Trigonometry*, pp. 125–127.[1] The formula (2) and the corresponding other formulas may be obtained by the use of the polar triangle with the aid of the theorem that the pole I' of the circumscribed circle for the triangle ABC is the pole of the inscribed circle for the polar triangle $A'B'C'$, and the pole I of the inscribed circle for ABC is the pole of the circumscribed circle for $A'B'C'$. This is easily proved geometrically. Formula (2) will be deduced in the desired manner, and from the result and (1) the formula (3) will be obtained. Suppose first that I' lies within the triangle ABC; then from the three isosceles triangles with the common vertex I', $A=\beta+\gamma$, $B=\gamma+\alpha$. $C=\alpha+\beta$, $S=\alpha+\beta+\gamma$. Hence $\alpha=S-A$, $\beta=S-B$, $\gamma=S-C$. Then from two of the isosceles triangles we have

$$\cos(S-A) = \tan(a/2)\cot P, \qquad \cos(S-B) = \tan(b/2)\cot P,$$

$$\frac{\cos(S-A)}{\cos(S-B)} = \frac{\tan(a/2)}{\tan(b/2)}. \tag{1}$$

If I' lies outside and on the side of BC opposite to A we have merely to replace α by $-\alpha$. The formulas (1) are then true in both cases. Now consider the triangle $\bar{A}\bar{B}\bar{C}$ which with ABC makes up the lune of

[1] The same process, essentially, is given by E. von Szücs, *Ebene und sphärische Trigonometrie auf ganz neuer Grundlage*, Zeitschrift für Mathematik und Naturwissenschaftlichen Unterricht, vol. 42 (1911), pp. 529–533.

angle A. For this triangle $\bar{A}=A$, $\bar{B}=180°-B$, $\bar{C}=180°-C$, $\bar{a}=a$, $\bar{b}=180°-b$, $\bar{S}=180°-S+A$, $\bar{S}-\bar{A}=180°-S$, $\bar{S}-\bar{B}=S-C$. Hence the last formula in (1) applied to this triangle gives

$$\frac{\cos S}{\cos (S-C)} = -\frac{\tan (a/2)}{\cot (b/2)}; \tag{2}$$

and from (1) and (2) we have at once

$$\tan^2 (a/2) = -\frac{\cos S \cos (S-A)}{\cos (S-B)\cos (S-C)}, \quad \cot^2 P = -\sec S \cos (S-A)\cos (S-B)\cos (S-C). \tag{3}$$

From the first formula above we obtain by two trigonometric reductions

$$\sec^2 (a/2) = \sin B \sin C/[\cos (S-B)\cos (S-C)].$$

Then, since $\sin^2 a = 4\tan^2(a/2)/\sec^4(a/2)$, we obtain

$$\sin^2 a = 4\cos^2 S \cot^2 P/\sin^2 B \sin^2 C.$$

Since $90°<S<270°$, we have

$$\frac{\sin a}{\sin A} = -\frac{2\cos S \cot P}{\sin A \sin B \sin C}.$$

This proves the law of sines. The corresponding formula is

$$\frac{\sin A}{\sin a} = \frac{2\sin s \tan p}{\sin a \sin b \sin c},$$

and the two combined give the result, after using the law of sines,

$$\frac{\sin A}{\sin a} = -\frac{\cos S \cot P}{\sin s \tan p}.$$

2686 [1918, 118]. Proposed by Edwin R. Smith, Iowa State College.

Given the difference equation,

$$u(x+1) - u(x) = \log\left(1 - \frac{x}{sp}\right) - \log\left(1 + \frac{x+1}{sq}\right),$$

where s is a positive integer, p and q are proper fractions such that $p+q=1$, and x is small compared to sp and sq. Determine $T(x)$, if $u(x)=\log T(x)$.

Solution by B. P. Hoover, Carnegie Institute of Technology

Consider the difference equation obtained by differentiating the given equation with regard to x. We have

$$u'(x+1) - u'(x) = -1/(sp-x) - 1/(sq+x+1),$$

where we notice that if the terms of the second member are integrated between the limits 0 and x and between -1 and x, respectively, we get the terms in the right member of the original equation.

A solution of the equation $F(x+1)-F(x)=1/x$, commonly called $\psi(x)$, is the following infinite series

$$F(x) = \sum_{i=0}^{\infty}\left(\frac{1}{i+1} - \frac{1}{x+i}\right)^{1}$$

Let $f(x)$ and $\phi(x)$ be solutions of $f(x+1)-f(x)=-1/(sp-x)$ and $\phi(x+1)-\phi(x)=1/(sq+x+1)$, respectively. Replacing x by $sp-x$ in $F(x+1)-F(x)=1/x$, we get $F(sp-x+1)-F(sp-x)=1/(sp-x)$, or, $F(sp-x)-F(sp-x+1)=-1/(sp-x)$. Hence we may take $F(sp-x+1)$ as our $f(x)$ and write

$$f(x) = \sum_{i=0}^{\infty}\left\{\frac{1}{i+1} - \frac{1}{sp-x+1+i}\right\}.$$

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[1] See, e.g., N. E. Nörlund's *Differenzenrechnung*, p. 102.

Similarly, we may take for our $\phi(x)$,

$$\phi(x) = \sum_{i=0}^{\infty} \left\{ \frac{1}{i+1} - \frac{1}{sq+x+1+i} \right\}.$$

These series converge uniformly with regard to x in the common interval $-sq-1+\epsilon_1 < x < sp+1-\epsilon_2$, where ϵ_1 and ϵ_2 are any positive numbers such that $-sq-1+\epsilon_1 < sp+1-\epsilon_2$. Hence, integrating term by term the series for $f(x)$ between the limits 0 and x, and the series for $\phi(x)$ between the limits -1 and x, and subtracting the latter from the former resulting series, [since we have taken $u'(x)=f(x)-\phi(x)$], we are led to the following series, which we denote by $u_1(x)$, which is valid in the same interval as before:

$$u_1(x) = \sum_{i=0}^{\infty} \left\{ -\frac{1}{i+1} + \log\left(\frac{sp-x+1+i}{sp+1+i} \cdot \frac{sq+x+1+i}{sq+i} \right) \right\}.$$

The difference, $u_1(x+1)-u_1(x)$, should differ from the right member of the first given equation by a constant. This difference is easily calculated by direct substitution and found to be $\log(p/q)$. Hence, we have only to add to $u_1(x)$ the quantity $-x \log(p/q)$ to have a solution of the first given equation. Denoting this solution by $u(x)$, we write

$$u(x) = -\left[x \log \frac{p}{q} + \sum_{i=0}^{\infty} \left\{ \frac{1}{i+1} - \log\left(\frac{sp-x+1+i}{sp+1+i} \cdot \frac{sq+x+1+i}{sq+i} \right) \right\} \right].$$

Finally, if $u(x)=\log T(x)$, we have $T(x)=\exp u(x)$, where the series given above replaces $u(x)$.

2891 [1921, 184]. Proposed by D. F. Barrow, Philomath, Ga.

Let A', A'', A''', and P denote, respectively, the vertices of a triangle and any point in its plane; and let P', P'', P''', denote the feet of the perpendiculars from P upon the sides opposite A', A'', and A'''. Now suppose each of the lines PP', PP'', and PP''' to revolve about P through an angle α; and let P_α', P_α'' and P_α''' denote the intersections of this new triad of lines with the corresponding sides of the triangle. As α varies, find the envelope of the variable circle through P_α', P_α'' and P_α'''.

Solution by Roscoe Woods, University of Iowa

There is no loss of generality if P be taken at the origin. Let $x \cos\theta_i + y \sin\theta_i = p_i$ $(i=1, 2, 3)$ be the equations of the sides of the triangle opposite the vertices A', A'' and A''' respectively. The coordinates of P' and P_α' are

$$(p_1 \cos\theta_1,\ p_1 \sin\theta_1) \quad \text{and} \quad [p_1 \cos(\theta_1+\alpha) \sec\alpha,\ p_1 \sin(\theta_1+\alpha) \sec\alpha],$$

respectively. By symmetry, we may write the coordinates of the points P'', P''', P_α'', and P_α'''. The equation of the variable circle through the points P_α', P_α'', and P_α''' is readily found to be of the following form when arranged so as to exhibit the parameter α,

$$\cos 2\alpha[A(x^2+y^2) + Bx - Cy] + \sin 2\alpha[Cx + By] + A(x^2+y^2) + Bx - Cy - D = 0,$$

where

$$A = \sum p_1 p_2 \sin(\theta_2 - \theta_1), \qquad B = \sum p_1 p_2 (p_2 \sin\theta_1 - p_1 \sin\theta_2),$$
$$C = \sum p_1 p_2 (p_2 \cos\theta_1 - p_1 \cos\theta_2), \quad D = 2 p_1 p_2 p_3 \sum p_3 \sin(\theta_2 - \theta_1),$$

and where the summation signs cover the cyclic advance of the subscripts 1, 2, and 3.

Let us note that $D=0$ is the condition that the variable circle pass through the origin, P. If $A=0$ the circle degenerates into a straight line for all α's. This line through the points P_α', P_α'', and P_α''' is known as Simson's generalised line, since the lines PP_α', PP_α'', and PP_α''' make equal angles with the sides of the given triangle. It is known that P lies on the circumcircle of the triangle $A'A''A'''$ in this case.

Using the regular method of obtaining the envelope, we find that it consists of the following parts

$$[A(x^2+y^2) + Bx - Cy]^2 + (Cx + By)^2 = 0,$$
$$2D[A(x^2+y^2) + Bx - Cy] + (Cx + By)^2 - D^2 = 0.$$

The only real point of the first is the origin and evidently this is extraneous. The second factor is a

conic which becomes a repeated line when $D=0$ and a parabola when $A=0$, this parabola being the envelope of all the generalised Simson lines associated with a given point P on the circumcircle of the triangle $A'A''A'''$.

2918 [1921, 327]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find planes which cut four given lines in four concyclic points.

Solution by Roscoe Woods, University of Iowa

In a space of three dimensions, a quadric surface may be defined as the aggregate of all points on all the transversals that can be drawn to three skew lines. See H. F. Baker, *Principles of Geometry*, Vol. 3, p. 1 ff. It is clear that each of these lines lies on the quadric. Let $l_i(i=1, 2, 3, 4)$ be the four given lines in space and choose any three of them, say $l_k(k=1, 2, 3)$ to determine a quadric Q as outlined above. The line l_4 cuts this quadric in two points, L and M, which can be real and distinct, coincident, or imaginary. It could happen that l_4 would lie entirely on the quadric.

The sections of a quadric by a system of parallel planes are conics, similar and similarly placed. Hence if one of the conics be a circle, all the sections made by the planes parallel to this one are circles. Now through each real finite point in space there pass six planes which intersect a given non-composite, non-spherical quadric in circles. Therefore, from each of the six systems of parallel planes so found there are two planes that cut the four lines $l_i(i=1, 2, 3, 4)$ in four concyclic points, namely, those determined by the points L and M. For the reality of these planes and the method of determining them, one should consult such a book as Snyder and Sisam, *Analytic Geometry of Space*, p. 96 ff. Because of the complexity of the problem, no attempt has been made to write down the solution for four general lines or to discuss any of the special cases that might arise, except to mention the fact that when l_4 lies entirely on the quadric Q all the planes of the six systems cut the four given lines in concyclic points.

Note by Otto Dunkel, Washington University

Let us assume that three of the straight lines l_1, l_2, l_3 do not lie in the same plane, and also that D_1 and D_2 are two points on l_1 and l_2, respectively, such that D_1D_2 does not cut and is not parallel to l_3. Take D_1D_2 as the x-axis of rectangular space coördinates with the origin at its mid-point, and the plane through D_1D_2 parallel to l_3 as the xz-plane. Let l_3 have the inclination γ to the xy-plane, let its trace in this plane be (a, b), and set $OD_1=d$, $OD_2=-d$. Then the circles through D_1 and D_2 and points on l_3 lie on the surface whose equation is

$$by(x^2+y^2+z^2-d^2)+d^2y^2-(ay+b\cot\gamma z)^2-b^2(y^2+z^2)=0.$$

Hence the line l_4 cuts this surface in one or three real points, and these points determine one or three real circles which satisfy the conditions of the problem. One or more of these circles may, however, degenerate into a straight line. Since D_1 and D_2 have each one degree of freedom it appears that there are an infinite number of constructions.

Particular constructions may be obtained by considering l_1, l_2, l_3 as three skew lines and as generators of the quadric surface containing the straight lines which touch them. If l_4 cuts this quadric in a point, the two real circular sections of the quadric through this point give two of the required circles. With the four lines we have in general four such quadrics. The quadric surface, however, is only a very special case of the surfaces containing l_1, l_2, l_3 such that all of its sections parallel to an arbitrarily given plane are circles. This more general surface is of the fourth order which degenerates into the quadric for special orientations of the given plane.

Also solved by Rufus Crane.

2977 [1922, 270]. Proposed by Florence P. Lewis, Goucher College.

A point moves in such a way that its polars with respect to two given conics intersect at right angles. Prove that the locus of this point of intersection is a rational quartic curve through the circular points and find its double points.

Solution by N. A. Court, University of Oklahoma

The point of intersection P' of the polars p_1, p_2 of the point P with respect to two given conics

(C_1), (C_2) is the conjugate of P with respect to both conics, hence P' corresponds to P in a quadratic transformation the singular points of which are the vertices of the triangle ABC which is self-polar to both (C_1) and (C_2).[1]

If the lines p_1, p_2 are perpendicular to each other, then the locus of P is, in general, a conic (C), as has been shown by the present writer some years ago. (See this MONTHLY, vol. 23, 1916, p. 255). Consequently the locus of P' is a quartic curve (Q) admitting the points A, B, C as double points.

Let i_1, i_2 be the polars of I, one of the circular points at infinity, with respect to (C_1), (C_2). The two polars of the point $H=(i_1 i_2)$ with respect to these two conics will pass through I, i.e., these two lines are parallel isotropic lines; but two such lines are also perpendicular to each other; hence H belongs to the conic (C),and therefore the conjugate I of H with respect to (C_1), (C_2) lies on (Q). Similarly for the other cyclical point I'. Thus (Q) passes through the circular points of the plane.

Also solved by J. H. NEELLEY.

3246 [1927, 156]. Proposed by Daniel Kreth, Wellman, Iowa.

From a point within a square, lines drawn to three of the vertices are 30, 40, and 50. Required the side of the square. A geometrical solution is desired.

SOLUTION BY CHARLES B. BUERGER, Pittsburg, Pennsylvania

The distances to three consecutive vertices are taken in the order $a=40$, $b=30$, $c=50$. Lay off any convenient length OE and the equal length OF at right angles to OE. Let G and G' divide OE internally and externally in the same ratio $OG/GE=b/a$. Then the circle with the diameter $G'G$ is the locus of all points P such that $OP/PE=b/a$. In a similar manner let H and H' divide OF internally and externally in the same ratio $OH/HF=b/c$, and construct the circle with the diameter $H'H$. Let J and M be the points of intersection of the two circles. Then $JE:JO:JF=a:b:c$. Lay off on OJ the length $OK=b$, and draw KL parallel to JE cutting OE or its prolongation in L. Then OL is a side of the required square. A second solution is given by a similar construction applied to M. In the case of the numerical example one square will contain the point and the other will not.

NOTE BY OTTO DUNKEL, Washington University

An analytical solution of this problem has already been printed 1928, 94. A geometric construction is easily read off from that solution. If the order is a, b, c as above (in the previous solution the order of the letters is b, a, c), then a right triangle $Q'BQ$ is constructed with the right angle at B so that $Q'B=QB=b$, and on the hypothenuse $Q'Q=2^{1/2}b$ the triangle $Q'AQ$ is constructed with the side $AQ=a$, $Q'A=c$. Then $BA=s$ is a side of the required square. There are in general two constructions for s. We have merely to rotate about B the triangle $BQ'A$ through 90° so that Q' falls on Q, and then A falls at a vertex C of the square, etc. As pointed out in that solution, if Q is inside the completed square, then s, its side, is the longer of the two lengths determined by the construction. It is easily seen that there are real constructions when the triangle QAQ' exists, that is if $0 \leqq a-c \leqq 2^{1/2}b \leqq a+c$, since we can assume without loss of generality that $a \geqq c$. But it does not follow that the larger square will always contain its point. Moreover there may be more than two solutions if we are allowed to take the lengths a, b, c in any order. In the numerical problem, the order may be taken as 50, 30, 40 or as 50, 40, 30. In each case there is a pair of solutions one of which gives a square with the point inside the square. But no real solution is given by the order 40, 50, 30. The analytical determination of s is simpler without the use of the complicated formula for the area of a quadrilateral which was used in the first solution. For, if we use two sides of the square as coördinate axes, the coördinates of the required point (x, y) are given by $2xs=s^2+b^2-a^2$, $2ys=s^2+b^2-c^2$, $x^2+y^2=b^2$. It then follows that $2s^2=a^2+c^2 \pm 4S$, where S is the area of the triangle with the sides a, c, $2^{1/2}b$. A necessary and sufficient condition that the point lies in the square is that s^2 be greater than or equal to the larger of the absolute values of b^2-c^2 and b^2-a^2, and that a, b, c satisfy the conditions for a real triangle as above. The numerical example $a=37$, $b=16$, $c=16$ gives a case for which neither square contains its point. For $a^2-b^2=1113$; $c^2-b^2=0$; $s=31.85, 24.71$; $s^2=1014.42, 610.58$. In

[1] Ch. Michel, *Compléments de Géométrie Moderne*, Vuibert, Paris, 1926, p. 53; Hilda Hudson, *Cremona Transformations*, Cambridge University Press, 1927, p. 33.

this test the value of s^2 must first be computed. An examination of the figure used in the construction gives another form of test which is rather complicated in its statement. The order is a, b, c and $a \geqq c$ as before. There are two cases according as $b \leqq 2^{1/2}c$ or $b > 2^{1/2}c$. Then necessary and sufficient conditions that the point lies within or on the boundary of the larger square are furnished in the first case by the inequalities

$$(2^{1/2}b - c)^2 \leqq a^2 \leqq b^2 + (b+c)^2, \quad b \leqq 2^{1/2}c.$$

In the second case the following inequalities must be satisfied

$$2b^2 - c^2 - 2c(b^2 - c^2)^{1/2} \leqq a^2 \leqq b^2 + (b+c)^2, \quad b > 2^{1/2}c.$$

3269 [1927, 335]. Proposed by J. Rosenbaum, Milford, Conn.

On the sides A_1B_1, B_1C_1, C_1A_1, of a triangle, $A_1B_1C_1$, the points A_2, B_2, C_2 are taken so that

$$A_1A_2 : A_2B_1 = B_1B_2 : B_2C_1 = C_1C_2 : C_2A_1 = \lambda,$$

thus forming a second triangle $A_2B_2C_2$. In this manner, by repeating the operation on the successive triangles a sequence of triangles $T_1, T_2, \cdots$, is obtained. Find under what conditions will (a) T_n be similar to T_1; (b) the shape of T_n approach a limit as n becomes infinite. (c) Find the point to which the three vertices, A_n, B_n, C_n converge when λ is a positive, proper fraction.

Solution by Paul Wernicke, Washington D. C.

For convenience, I call the given triangle $A_0B_0C_0$ and use barycentric coordinates referred to it or, what is the same, vectors A, B, C from any origin outside its plane to its vertices A_0, B_0, C_0. Then, for the first derived triangle T_1, I put $\lambda = p/q = B_0A_1/A_1C_0 = C_0B_1/B_1A_0 = A_0C_1/C_1B_0$, etc., putting A_{n+1} on B_nC_n, etc. Also, $p+q=1$. The vertices of T_n are found to be

$$A_n = h_nA + k_nB + l_nC, \quad B_n = l_nA + h_nB + k_nC, \quad C_n = k_nA + l_nB + h_nC,$$

where h_n, as well as k_n and l_n, is the result of adding every third term in the expansion $\sum_\nu \binom{n}{\nu} p^{n-\nu}q^\nu$ ($\nu = 0 \cdots n$) of $(p+q)^n$, h_n beginning with the first term when $n \equiv 0$, with the third when $n \equiv 1$, and with the second when $n \equiv 2 \pmod 3$; k_n begins with the second term when $n \equiv 0$, with the first when $n \equiv 1$ and with the third when $n \equiv 2$. l_n begins with the third term when $n \equiv 0$, with the second when $n \equiv 1$ and with the first when $n \equiv 2$.

Every T_n has the same centroid as T_0, viz. $(A+B+C)/3$, as the three coefficients used add to $(p+q)^n = 1$. This is also quite evident mechanically. The ratio of areas, Δ_1/Δ_0, is $p^2+q^2 = p^2 - pq + q^2 = 1 - 3pq = 1 - 3\lambda(\lambda+1)^{-2}$ and the ratio Δ_n/Δ_0 is

$$r_n^2 = (1 - 3pq)^n = \begin{vmatrix} h_n & k_n & l_n \\ l_n & h_n & k_n \\ k_n & l_n & h_n \end{vmatrix} = \begin{vmatrix} 1 & k_n & l_n \\ 1 & h_n & k_n \\ 1 & l_n & h_n \end{vmatrix} = \left\{ \begin{matrix} h_n^2 + k_n^2 + l_n^2 \\ - k_nl_n - l_nh_n - h_nk_n \end{matrix} \right\}.$$

The vector side B_nC_n of T_n is

$$C_n - B_n = a_n = h_n(C - B) + k_n(A - C) + l_n(B - A) = h_na + k_nb + l_nc, \text{ whence}$$

$$\begin{aligned} a_n^2 &= h^2a^2 + k^2b^2 + l^2c^2 - 2klbc \cdot \cos A - 2lhca \cdot \cos B - 2hkab \cos C \\ &= (h^2 + kl - lh - hk)a^2 + (k^2 + lh - hk - kl)b^2 + (l^2 + hk - kl - lh)c^2, \end{aligned}$$

where h, k, l are supposed to be written with the subscript n. We have similar expressions for b_n^2 and c_n^2.

We find that $a_n^2 + b_n^2 + c_n^2 = r_n^2(a^2+b^2+c^2)$, the meaning of the summation Σ being readily discernible. It is intuitively evident that only if $\sum |a_n| = r_n \sum a$ can there be (straight) similarity (i.e. $A_nB_nC_n \sim ABC$ and not to ACB or another permutation). If $A_nB_nC_n$ were similar to ACB, then $A_{2n}B_{2n}C_{2n} \sim A_nC_nB_n \sim ABC$ and if $A_nB_nC_n \sim BCA$, then $A_{3n}B_{3n}C_{3n} \sim ABC$. Therefore, if any similarity were obtained, then $A_{6n}B_{6n}C_{6n} \sim ABC$.

From the above, the answers to the three questions become:

(a) There can be congruence if $\lambda = 0$ or if $\lambda = \infty$, only, and similarity only in the two trivial cases, (1) $\lambda = 1$ or $p = q = 1/2$ and (2) $a = b = c$. In both cases all triangles T_n are similar (straight).

(b) When $\lambda > 0$, T_n approaches the equilateral shape as $n \to \infty$, while when $\lambda < 0$, T_n approaches a straight line (at infinity).

(c) When $\lambda>0$, since all T_n have the same centroid and the area diminishes without bounds, this centroid is the point towards which the vertices converge. In this connection it is of interest to note that the three triangles cut off by the sides of T_{n+1} and each having one of the vertices A_n, B_n, C_n, have each pq times the area of T_n.

3272 [1927, 380]. Proposed by L. M. Berkeley, New York City.

Solve the differential equation,

$$(\tan x + m \sin y)dy = (\sin y - m \tan x) \cos y dx,$$

where m is a constant.

I. Solution by John P. Dalton, University of the Witwatersrand, Johannesburg, S. A.

Two solutions of this problem are given in the article, *On the integration of a certain type of first order differential equation*, in this Monthly, vol. 35 (1928), pp. 189–191.

II. Solution by Paul Wernicke, Washington, D. C.

Multiply the given equation by $\cos x$. It may then be written:

$$\sin y \cdot \cos y \cdot \cos x \cdot dx - \sin x \cdot dy - m(\cos y \cdot \sin x \cdot dx + \cos x \cdot \sin y \cdot dy) = 0.$$

The parenthesis multiplied by $-m$ is $-d$ ($\cos x \cdot \cos y$). Put $\cos x \cdot \cos y = u$. The first two terms in (1) do not constitute a complete differential, but give $-d(\tan y/\sin x)$ when multiplied by the integrating factor $(\sin x \cdot \cos y)^{-2}$. Put $\tan y/\sin x = v$. Then $(\sin x \cdot \cos y)^2 = (1-u^2)/(1+v^2)$, so that (1) becomes

$$(1 + v^2)^{-1}dv + m(u^2 - 1)^{-1}du = 0,$$

giving $\tan^{-1}v + (m/2) \log (1-u)(1+u)^{-1} = \text{constant}$, in which the values for u and v are to be substituted.

3273 [1927, 381]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The three points of intersection of the symmedians of a triangle with its circumcircle determine a triangle whose circles of Apollonius are identical with those of the given triangle.

Solution by Paul Wernicke, Washington, D. C.

The Apollonian circle (E) through A for the triangle ABC has its center E on BC, and it cuts the circumcircle (K) of ABC orthogonally at A and at a second point denoted by A'. The polar of E with respect to (K) is therefore AA', and thus the pencil $A\ (BA'\ CE)$ is harmonic. Hence AA' is the symmedian through A as is well known. In a similar manner the corresponding points B' and C' are located. It is also well known that BB' and CC' meet in a point L on AA'. According to the definition of a polar, BC and $B'C'$ meet at a point on the polar of L; this point must be E, the intersection of BC and the polar of L, so that E is on the line $B'C'$. It was shown above that the Apollonian circle (E) of the triangle ABC is characterized by the facts that it intersects the circumcircle orthogonally at A and has center on BC. Since the same circle has the corresponding property with respect to the triangle $A'B'C'$, it follows that these two triangles have the Apollonian circle (E), and similarly the other two Apollonian circles, in common.

Also solved by William Hoover and J. H. Neelley.

3274 [1927, 381]. Proposed by C. N. Mills, Normal, Illinois.

The axes of three mutually perpendicular right circular cylinders intersect in a common point. If the radii are equal, find the common volume.

Solution by Emma M. Gibson, Senior High School, Springfield, Missouri

Let the equations of the cylinders whose axes are the x, y, and z axes of coordinates be, respectively,

$y^2+z^2=R^2$, $x^2+z^2=R^2$, and $x^2+y^2=R^2$. The volume common to two of the cylinders, say, $x^2+z^2=R^2$ and $y^2+z^2=R^2$ is $16R^3/3$. The third cylinder cuts off from this volume in the first octant a volume,

$$v = 2\int_{2^{-1/2}R}^{R} dx \int_{(R^2-x^2)^{1/2}}^{x} dy \int_0^{(R^2-y^2)^{1/2}} dz = \frac{R^3}{3}(3\cdot 2^{1/2}-4).$$

Hence, the volume common to the three cylinders is

$$(16/3)R^3 - (8/3)R^3(3\cdot 2^{1/2} - 4) = 8R^3(2 - 2^{1/2}).$$

Also solved by J. M. Earl, Michael Goldberg, Enrique Linares, Jr., J. B. Reynolds, and Paul Wernicke.

3275 [1927, 381]. Proposed by J. A. Shohat, University of Michigan.

Solve the differential equation,

$$xydy/dx = y^2 + a^2 - ax \pm [(y^2 + a^2 - ax)^2 - y^2(y^2 + a^2 - 2ax)]^{1/2}.$$

Solution by Frederic H. Miller, Cooper Union Institute of Technology, New York, N. Y.

The given equation reduces to

$$xydy/dx = y^2 + a^2 - ax \pm a[(x - a)^2 + y^2]^{1/2}.$$

Now introduce a new variable, $z=[(x-a)^2+y^2]^{1/2}$; then $xy\,dy/dx=xz\,dz/dx-x^2+ax$ and $y^2=z^2-(x-a)^2$. Discarding $z=0$, we get $x\,dz/dx=z\pm a$; whence $z\pm a=ex$,

or

$$[(x - a)^2 + y^2]^{1/2} \pm a = ex,$$

e being the constant of integration.

This last equation may be identified with that of a conic with focus at $(a, 0,)$ and with eccentricity e.

Also solved by H. H. Downing, J. D. Tamarkin and Paul Wernicke.

3277 [1927, 381]. Proposed by R. H. Sciobereti, Berkeley, California.

Find the motion of a weighing particle of mass 1 moving in a vertical plane and attracted by a point of this plane with a force k/r^2, k being a positive constant. Consider applications of the result to the two special cases defined by the following initial conditions: (*i*) at $t=t_0$, the coordinates of M are (0, b), $b>0$, and velocity, given by $v_0^2=2(gb^2+k)/b$, is parallel to the axis of x; (*ii*) at $t=t_0$, the coordinates of M are $(0, b)$, with $-(k/g)^{1/2}<b<0$, and $v_0^2=2(gb^2-k)/b$, with v_0 parallel to the axis of x. What can be said of the stability of the motion in these two particular cases?

Solution by the Proposer.

Let us take a system of rectangular axes having its origin at the center of attraction O, the axis of y being the vertical line through O and directed positively upward. The position of the particle M will be determined by its distances r and y from the origin and the axis of x respectively, or by a set of coordinates (u, v) defined by the two relations

$$r + y = 2u, \qquad r - y = 2v,$$

so that the curves $u=$ const. and $v=$const. represent a system of confocal parabolas, with focus at O. Now the equations of motion in finite form will be derived from a complete integral of the Hamilton-Jacobi partial differential equation. In order to form this equation we shall calculate the kinetic energy T, a quadratic function in u' and v', and the force function $U=kr^{-1}-gy$, or in terms of the coordinates (u, v), $U=k(u+v)^{-1}-g(u-v)$. Furthermore $2T=x'^2+y'^2$ which after a few algebraic transformations becomes $2T=(u+v)(u'^2u^{-1}+v'^2v^{-1})$, and since the time does not appear explicitly the Hamilton-Jacobi equation takes the form $H=T-U=h$, where H is a function of the quantities u, v, $p_1=\partial T/\partial u'$ and $p_2=\partial T/\partial v'$; hence replacing the variables p_1 and p_2 by W_u and W_v respectively, we shall find for the kinetic energy

$$2T = (u + v)^{-1}(uW_u^2 + vW_v^2),$$

and for the partial differential equation,

$$(u+v)^{-1}(uW_u^2+vW_v^2)=2k(u+v)^{-1}-2g(u-v)+2h.$$

It can be integrated by a separation of variables; that is, there exists a complete integral of the form $W=W_1(u)+W_2(v)$, since according to Stäckel's theorem, six functions $f_1(u)$, $f_2(u)$, $\phi_1(v)$, $\phi_2(v)$, $\psi_1(u)$ and $\psi_2(v)$ may be found such that

$$(1)\qquad u(u+v)^{-1}=\Delta^{-1}(\partial\Delta/\partial f_1),\qquad v(u+v)^{-1}=\Delta^{-1}(\partial\Delta/\partial\phi_1),$$

$$(2)\qquad 2U=2k(u+v)^{-1}-2g(u-v)=\Delta^{-1}[\psi_1(u)(\partial\Delta/\partial f_1)+\psi_2(v)(\partial\Delta/\partial\phi_1)],$$

where $\Delta=(f_1\phi_2-f_2\phi_1)$. Remembering that $\Delta=f_1(\partial\Delta/\partial f_1)+\phi_1(\partial\Delta/\partial\phi_1)$, we shall find with the help of (1) the relation $u+v=uf_1+v\phi_1$; hence the functions f_1 and ϕ_1 may be taken equal to unity. On the other hand ϕ_2 must be a function of v only, so that selecting for Δ the value $u^{-1}+v^{-1}$, it follows that $\phi_2=v^{-1}$ and $f_2=-u^{-1}$. As to the functions ψ_1 and ψ_2, they are readily determined from (2) and found to be $\psi_1=ku^{-1}-2gu$, $\psi_2=kv^{-1}+2gv$. The partial derivatives $W_u^2=(\partial W_1/\partial u)^2$; $W_v^2=(\partial W_2/\partial v)^2$ are given by the formulas

$$W_u^2=ku^{-1}-2gu+2h-2\alpha u^{-1},\qquad W_v^2=kv^{-1}+2gv+2h+2\alpha v^{-1},$$

from which the complete integral follows at once:

$$W=\int(ku^{-1}-2gu+2h-2\alpha u^{-1})^{1/2}du+\int(kv^{-1}+2gv+2h+2\alpha v^{-1})^{1/2}dv,$$

where α denotes an arbitrary constant. The equations of motion are therefore

$$\partial W/\partial h=\beta+t=\int F(u)du+\int G(v)dv,\qquad \partial W/\partial\alpha=\gamma=-\int F(u)du/u+\int G(v)dv/v,$$

where

$$F(u)=(ku^{-1}-2gu+2h-2\alpha u^{-1})^{-1/2},\ G(v)=(kv^{-1}+2gv+2h+2\alpha v^{-1})^{-1/2}.$$

From the initial conditions given in (*i*), $u_0=b$, $v_0=0$; hence $h=2gb$; and since $(\partial W/\partial u)_0=0$, it follows that $2\alpha=2gb^2+k$; therefore for this particular case

$$F(u)=u^{1/2}[-2g(u-b)^2]^{-1/2},\qquad G(v)=v^{1/2}[2g(v+b)^2+2k]^{-1/2}.$$

It is then easy to see that $u=b$ during the whole motion, while v varies in the same direction. The trajectory is the downward parabola $u=b$, the motion being stable, for if the initial conditions are altered by quantities of the first order, the function $u(k-2\alpha-2gu^2+2hu)$ which will vanish for two values of u infinitely near b will be positive for any value of u lying between these roots.

For the initial conditions (*ii*) we shall find in an exactly similar manner $u_0=0$, $v_0=-b$, $U_0=-kb^{-1}-gb^{-1}$; hence $h=2gb$, and since $(\partial W/\partial v)_0=0$, $2\alpha=2gb^2-k$, the functions $F(u)$ and $G(v)$ will reduce to $u^{1/2}[2k-2g(u-b)^2]^{-1/2}$ and $v^{1/2}[2g(v+b)^2]^{-1/2}$, rsepectively. v remains equal to $-b$, and u oscilates between the extreme values 0 and $b+(kg^{-1})^{1/2}$; the path is therefore an arc of the upward parabola $v=-b$, the oscillatory motion being stable for the same reason as before.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor J. W. Lasley, of the University of North Carolina, has been elected chairman of the mathematics section of the North Carolina Academy of Science, and Professor W. W. Elliott, of Duke University, secretary of that section.

Professor HARRY BATEMAN, of the California Institute of Technology, has been elected a fellow of the Royal Society of London.

Williams College has conferred the honorary degree of doctor of science on Professor W. A. BRATTON, of Whitman College.

Columbia University has conferred an honorary doctorate on President W. W. CAMPBELL, of the University of California, director of the Lick Observatory.

The University of Indiana has conferred an honorary degree on Professor J. A. MILLER, of Swarthmore College.

Princeton University has conferred an honorary degree of doctor of science on Professor R. A. MILLIKAN.

Professor D. E. SMITH, retired, of Columbia University, has returned from Europe where he has served as a visiting Carnegie professor of international relations. Representing the Carnegie Endowment for International Peace, he lectured at the University College, London, and at the Universities of Montpellier and Toulouse.

The next annual meeting of the National Council of Teachers of Mathematics will be held at the Hotel Statler, Cleveland, Ohio, on February 22–23, 1929. Fuller details will be announced later in the year.

The final printing of the Rhind Mathematical Papyrus is now in progress, which indicates that the binding may be completed early in the autumn. For a description of this forthcoming publication, see page 445 of the November, 1927 issue of this Monthly.

The Proceedings of the International Mathematical Congress of 1924, which have recently been published in two volumes, may be obtained from the University of Toronto Press. The price is five dollars plus postage to registered members of the Congress and ten dollars plus postage to others. Separate copies of the group photograph of the 230 members of the Congress may be obtained for 50 cents each.

The following graduate courses in mathematics are announced for the academic year 1928–29:

The University of Chicago—By Professer E. H. MOORE: General analysis, III, IV, V; Seminar on the foundations of general analysis, I, II, III. By Professor H. E. SLAUGHT: Differential equations; Definite integrals; Elliptic integrals. By Professor L. E. DICKSON: Advanced topics in algebra and the theory of numbers, I, II; Algebras and their arithmetics. By Professor G. A. BLISS: Calculus of variations, I, II. By Professor A. C. LUNN: Lattices and crystal groups; Theory of sound; Relativity; Mathematical theory of electro-

magnetism. By Professor W. D. MACMILLAN: Analytic mechanics, I, statics, II, dynamics; Celestial mechanics, I; Theory of the potential; Dynamics of rigid bodies. By Professor E. P. LANE: Conjugate sets; Metric differential geometry; Synthetic projective geometry; Projective differential geometry of higher space. By Professor M. I. LOGSDON: Functions of a complex variable; Elliptic functions; Algebraic functions with applications. By Professor L. M. GRAVES: Vectors, matrices, and quaternions; Modern theories of integration; Functions of lines. By Professor R. W. BARNARD: General analysis, I, II; Theory of numbers. By Professor WALTER BARTKY: Modern theories of differential equations, I, II. By Dr. B. W. JONES: Introduction to higher algebra; Theory of numbers. Reading and research courses are offered by Professors Moore and Barnard in the foundations of mathematics and general analysis, and by Professor Lunn in applied mathematics. Thesis work and supervision are offered by Professors Dickson and Barnard in algebra and the theory of numbers; by Professors Bliss and Graves in analysis; by Professor Lane in differential geometry; by Professor Logsdon in algebraic geometry; and by Professors MacMillan and Bartky in applied mathematics.

Assistant Professor C. R. ADAMS, of Brown University, has been promoted to an associate professorship of mathematics.

Dr. H. A. BENDER, formerly instructor in the University of Illinois, has been made assistant professor of mathematics at the University of Akron.

Associate Professor R. W. BRINK, of the College of Science, Literature, and the Arts of the University of Minnesota, has been promoted to a professorship of mathematics and has been appointed chairman of the department of mathematics for the year 1928–29.

Professor J. A. BULLARD has resigned his position at the United States Naval Academy to accept the professorship of mathematics in the college of engineering, University of Vermont, *vice* Professor EVAN THOMAS, recently retired.

Associate Professor ALAN D. CAMPBELL has been promoted to a full professorship of mathematics at Syracuse University.

Miss ELIZABETH CARLSON has been promoted to an assistant professorship of mathematics at the University of Minnesota.

Dr. M. G. CARMAN has been appointed head of the department of mathematics at Murray State Teachers College, Murray, Ky.

Associate Professor A. F. CARPENTER, of the University of Washington, has been promoted to a professorship of mathematics.

Mr. WILLIAM FITCH CHENEY, JR. has been promoted to an assistant professorship of mathematics at Tufts College.

Dr. N. B. CONKWRIGHT has been promoted to the rank of associate at the University of Iowa.

Assistant Professor LENNIE P. COPELAND has been promoted to an associate professorship at Wellesley College.

Assistant Professor C. H. CURRIER has been promoted to an associate professorship of mathematics at Brown University.

Assistant Professor W. E. EDINGTON has been promoted to an associate professorship of mathematics at Purdue University.

Dr. RAYMOND GARVER, of the University of Rochester, has been appointed assistant professor of mathematics at the University of California at Los Angeles.

Assistant Professor G. H. GRAVES has been promoted to an associate professorship of mathematics at Purdue University.

Dr. CHARLES C. GROVE is transferring from the Polytechnic Institute of Brooklyn, where he has been since Jan. 1, 1925, to the Willoughby Branch of the College of the City of New York.

Professor W. L. HART, of the University of Minnesota, has been granted a sabbatical leave of absence for the year 1928–29. He will spend most of the year on the Pacific coast.

Dr. VERN JAMES has been appointed head of the department of mathematics at Menlo Junior College.

Mr. F. C. JOHAH has been appointed assistant professor of mathematics at Adelbert College.

Dr. E. G. KELLER has been appointed assistant professor of mathematics at the University of Texas.

Assistant Professor CLARIBEL KENDALL, of the University of Colorado, has been promoted to an associate professorship of mathematics.

At Columbia University, Dr. B. O. KOOPMAN was recently promoted to an assistant professorship of mathematics, not to a full professorship as was erroneously announced in the June-July issue of this Monthly.

Associate Professor E. P. LANE has been promoted to a professorship of mathematics at the University of Chicago.

Dr. C. G. LATIMER, of the University of Kentucky, has been promoted to a professorship of mathematics.

Dr. R. G. LUBBEN has been appointed adjunct professor of mathematics at the University of Texas.

Associate Professor JOHN H. MCDONALD has been promoted to a professorship of mathematics at the University of California at Berkeley.

Dr. A. J. MARIA has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor R. W. MARRIOTT has been promoted to a professorship of mathematics at Swarthmore College.

Dr. G. M. MERRIMAN has been appointed assistant professor of mathematics at Grinnell College.

Professor CLIFFORD N. MILLS, of Illinois State Normal University, has been granted a leave of absence for 1928–29 for graduate study at the University of Michigan.

Assistant Professor D. S. MORSE has been promoted to an associate professorship of mathematics at Union College.

Assistant Professor MARSTON MORSE has been promoted to an associate professorship of mathematics at Harvard University.

Associate Professor F. D. MURNAGHAN, of Johns Hopkins University, has been promoted to a professorship of applied mathematics.

Dr. J. R. MUSSELMAN, of Johns Hopkins University, has been appointed professor of mathematics in the Woman's College of Western Reserve University.

Assistant Professor J. A. NORTHCOTT has been promoted to an associate professorship of mathematics at Columbia University.

Mr. H. S. POLLARD has been appointed to an assistant professorship of mathematics at Miami University.

Professor P. R. RIDER, of Washington University has been appointed Sterling Research Fellow at Yale University for the year 1928–29.

Dr. W. E. ROTH, of the University of Wisconsin, has been appointed professor in the university's extension division at Milwaukee.

Professor JOSEPH EUGENE ROWE has resigned his position as head of the department of mathematics and director of extension at the College of William and Mary to accept the presidency of the Clarkson Memorial Institute of Technology at Potsdam, N. Y.

Assistant Professor H. A. SIMMONS has been promoted to an associate professorship at Northwestern University.

Assistant Professor HERMAN L. SMITH has been promoted to an associate professorship of mathematics at Louisiana State University.

Dr. M. H. STONE has been promoted to an assistant professorship of mathematics at Harvard University.

Professor OSWALD VEBLEN, of Princeton University, has been appointed to a research professorship at that university.

At the University of Iowa, Dr. L. E. WARD has been promoted to an assistant professorship of mathematics.

Assistant Professor WARREN WEAVER, of the University of Wisconsin, has been promoted to a professorship of mathematics.

Dr. G. T. WHYBURN has been promoted to an adjunct professorship of mathematics at the University of Texas.

Professor W. M. WHYBURN, formerly of the Texas Technological College, and recently National Research Fellow, has been appointed assistant professor of mathematics at the University of California at Los Angeles.

The following appointments to instructorships in mathematics are announced:

University of Arkansas, Mr. D. P. RICHARDSON;
Brooklyn Polytechnic Institute, Mr. D. E. WHITFORD;
Bryn Mawr College, Mr. H. H. PIXLEY;
Cornell University, Mr. L. J. PARADISO;
Hunter College, Miss R. L. ANDERSON;
University of Illinois, Dr. BESSIE I. MILLER;
Johns Hopkins University, Dr. L. M. BLUMENTHAL;
Lafayette College, Mr. J. H. FITHIAN;
Lehigh University, Dr. H. B. HAMMATT, Messrs. KEELER and HUGH STANLEY, Dr. W. J. TRJITZINSKY, Mr. A. N. VAN ARNAM;
University of Michigan, Mr. F. S. BEALE;
University of Minnesota, Dr. J. M. EARL;
Oregon State Agricultural College, Miss MELVA K. THOMPSON;
University of Rochester, Miss ROSE A. WHELAN;
Sweet Briar College, Miss JULIA W. BOWER;
Williams College, Mr. H. L. DORWART;
Winthrop College, Miss ALICE A. GRANT;
University of Wisconsin, Mr. J. I. VASS.

Professor R. E. ALLARDICE, of Stanford University, died May 29, 1928, at the age of sixty-six.

Mr. PHILIP FITCH, Assistant principal of North High School, Denver, Colorado, died suddenly on July 23. He was secretary of the Rocky Mountain Section of the Mathematical Association of America.

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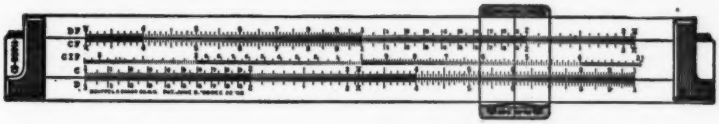

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December 28-29, 1928

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, April 28.

OHIO, Columbus, Ohio, April 5.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Dunham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-nine persons and two institutions have been elected to membership in the Association, on applications duly certified:

To Individual Membership

J. M. BARBOUR, A. M. (Dickinson). Asst. Prof. Music, Wells Coll., Aurora, N. Y.

W. V. BOND, Student of Math., Univ. of Göttingen, Germany.

G. C. BORDNER, A.M. (Franklin and Marshall), Prof., State Teachers Coll., Kutztown, Pa.

MABELLE C. DAME, Ph.D. (Chicago). Pr f. Math. and Physics, Brenau Coll., Gainesville, Ga.

MARGUERITE D. DARKOW, Ph.D. (Chicago). Asst. Prof., Penn. State Coll., State College, Pa.

F. W. DOERMANN, Ph.D. (Vienna). Instr., Physics, New York Univ., New York, N. Y.

P. S. DWYER, A.M. (Penn. State). Asst. Prof., Antioch Coll., Yellow Springs, Ohio.

C. M. ERIKSON, A.M. (Michigan). Part-time instr., Michigan State Normal Coll., Ypsilanti, Mich.

K. G. FULLER, A.M. (Nebraska). Instr., Northwestern Univ., Evanston, Ill

LOIS W. GRIFFITHS, Ph.D. (Chicago). Instr., Northwestern Univ., Evanston, Ill.

R. J. HANNELLY, M.S. (Iowa). Head of Dept., Junior Coll., Phoenix, Ariz.

B. R. ICAMEN, A.B. (Occidental), B.S.C.E. (Southern California). Acting Head of Dept., Silliman Inst., Maasin, Leyte, P. I.

E. E. INGALLS, Ph.B., M.F. (Yale). Asst. Prof., Iowa Wesleyan Coll., Mount Pleasant, Ia.

H. F. JOHNSON, A.B. (Cincinnati). Fellow, Univ. of Cincinnati, Cincinnati, Ohio.

HERMAN KARNOW, A.B. (Colorado). Asst. instr., Univ. of Colorado, Boulder, Colo.

ALICE B. KELSEY, A.M. (Illinois). Instr., Southern Illinois Normal Univ., Carbondale, Ill.

CAROLINE A. LESTER, A.M. (Cornell). Teacher, Low and Heywood School, Stamford, Conn.

G. M. MERRIMAN, Ph.D. (Cincinnati). Asst. Prof., Grinnell Coll., Grinnell, Ia.

R. M. PINKERTON, B.S. (Bradley Polytech.) Rushville, Ill.

G. A. PLIMPTON, A.B. (Amherst), LL.D. (Rochester, Richmond), L.H.D. (New York Univ.) Publisher, Ginn and Co., 70 Fifth Ave., New York, N. Y.

GANESH PRASAD, M.A. (Cambridge), D.Sc. (Allahabad). Prof., Higher Math., Calcutta Univ. Calcutta, India.

J. A. REISING, Teacher, Central High School and Indiana Univ. Extension, Fort Wayne, Ind.

L. J. ROTE, Practicing Attorney, Denver, Colo.

T. P. SCOTT, A.M. (Mississippi). Prof., Vice Pres. and Head of Dept., State Teachers Coll., Hattiesburg, Miss.

C. C. SHERMAN, M.S. (Iowa). Grad. Student, Univ. of Iowa, Iowa City, Ia.

F. E. SMITH, A.M. (Catholic Univ.) Asst. and Grad. Student, Catholic Univ. of America, Washington, D. C.

W. O. SWINYARD, B.S. (Utah Agric.) Instr., High School, Malad, Idaho.

MILDRED G. THIESMEYER, A.B. (Hunter). Tutor, Hunter Coll., New York, N. Y

E. C. WARREN, B.S. (Mass. Inst. of Tech.) Instr., Colby Coll., Waterville, Me.

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THE FIFTH ANNUAL MEETING OF THE TEXAS SECTION

The fifth annual meeting of the Texas Section of the Mathematical Association of America was held at Texas Agricultural and Mechanical College, College Station, Texas on Saturday, January 28, 1928. For the first time since its organization, this section held an independent meeting of its own, all previous meetings having been held in conjunction with the Mathematics Section of the Texas State Teachers Association at the time of its regular annual meeting at Thanksgiving.